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On the commutative quotient of Fomin–Kirillov algebras

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ABSTRACT

The Fomin–Kirillov algebra $\mathcal{E}_n$ is a noncommutative algebra with a generator for each edge of the complete graph on n vertices. For any graph G on n vertices, let $\mathcal{E}_G$ be the subalgebra of $\mathcal{E}_n$ generated by the edges in G. We show that the commutative quotient of $\mathcal{E}_G$ is isomorphic to the Orlik–Terao algebra of G. As a consequence, the Hilbert series of this quotient is given by $(-t)^n\chi_G(-t^{-1})$, where χ_G is the chromatic polynomial of G. We also give a reduction algorithm for the graded components of $\mathcal{E}_G$ that do not vanish in the commutative quotient and show that their structure is described by the combinatorics of noncrossing forests.

1. Introduction

Fomin and Kirillov [3] introduced a noncommutative algebra $\mathcal{E}_n$ for the purpose of understanding the generalized Littlewood–Richardson problem of computing intersection numbers in the flag variety. Since then, this algebra and its generalizations have been studied extensively elsewhere: see, for instance, [1,2,4–9,11,12,15,17]. Unfortunately, many key facts about $\mathcal{E}_n$, such as its Hilbert series, remain unknown for most values of n. In order to further study $\mathcal{E}_n$, the authors of [2] describe a subalgebra $\mathcal{E}_G \subseteq \mathcal{E}_n$ for any graph G on n vertices and discuss its properties.

Fomin and Kirillov [3] attribute to Varchenko the observation that the commutative quotient of $\mathcal{E}_n$ has dimension $n!$ and Hilbert series

$$(1+t)(1+2t)\cdots(1+(n-1)t).$$

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In this paper, we will show that an analogous result holds for the commutative quotient $\mathcal{E}_G^{ab}$ of $\mathcal{E}_G$. Specifically, we will show that $\mathcal{E}_G^{ab}$ is isomorphic to the Orlik–Terao algebra U_G (defined in [14]), which is known to have Hilbert series $(-t)^n \chi_G(-t^{-1})$. This resolves a conjecture of Kirillov [5].

We also discuss the structure of the graded components of $\mathcal{E}_G$ that do not vanish in the commutative quotient. This can be thought of as giving a sort of noncommutative analogue of the Orlik–Terao algebra. We show that one can describe a basis for these components in terms of certain noncrossing forests. We also give a reduction algorithm for these components and show that this reduction gives a unique expression of any element in terms of basis elements (independent of the choices made during the reduction). This reduction is similar to the reductions in subdivision algebras given by Mészáros in [10].

We begin in Section 2 with the proof that $\mathcal{E}_G^{ab}$ is isomorphic to the Orlik–Terao algebra U_G. In Section 3, we discuss the combinatorics of noncrossing trees and their relationship to the structure of certain graded components of $\mathcal{E}_G$. Finally, we conclude in Section 4 with some brief final remarks and observations.

2. Fomin–Kirillov algebras

We begin with some preliminaries about the Fomin–Kirillov algebras $\mathcal{E}_G$. For more information, see [2,3].

Definition 2.1. The *Fomin–Kirillov algebra* $\mathcal{E}_n$ is the quadratic algebra (say, over $\mathbf{Q}$) with generators $x_{ij} = -x_{ji}$ for $1 \le i < j \le n$ with the following relations:

- $x_{ij}^2 = 0$ for distinct i, j;
- $x_{ij}x_{kl} = x_{kl}x_{ij}$ for distinct i, j, k, l;
- $x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0$ for distinct i, j, k.

For any graph G with vertex set $[n]$, the *Fomin–Kirillov algebra* $\mathcal{E}_G$ of G is the subalgebra of $\mathcal{E}_n$ generated by x_{ij} for all edges $\overline{ij}$ in G.

In particular, $\mathcal{E}_n = \mathcal{E}_{K_n}$, where K_n is the complete graph with vertex set $[n]$. Note that since the set of relations of $\mathcal{E}_n$ is fixed by relabelings of the vertex set, the structure of $\mathcal{E}_G$ depends only on the structure of the graph G up to isomorphism.

Typically $\mathcal{E}_G$ will have minimal relations that are not quadratic. The most important relations for our purposes will be the following, proved in [2] by a straightforward induction.

Proposition 2.2 ([2]). *For distinct $i_1, i_2, \ldots, i_m \in [n]$,*

$$x_{i_1i_2}x_{i_2i_3}\cdots x_{i_{m-1}i_m} + x_{i_2i_3}x_{i_3i_4}\cdots x_{i_{m-1}i_m}x_{i_mi_1} + \cdots + x_{i_mi_1}x_{i_1i_2}\cdots x_{i_{m-2}i_{m-1}} = 0.$$

When $m = 3$, this is the third quadratic relation in the definition of $\mathcal{E}_n$.

2.1. Grading

The algebras $\mathcal{E}_G$ have three natural gradings:

- Any monomial $P \in \mathcal{E}_G$ has the usual notion of degree, denoted $d(P)$.
- There is a grading with respect to the symmetric group $\mathfrak{S}_n$: we define the $\mathfrak{S}_n$-degree of x_{ij} to be the transposition $(i\ j) \in \mathfrak{S}_n$ and extend to all monomials by multiplicativity. Denote the $\mathfrak{S}_n$-degree of P by σ_P.
- For any monomial $P \in \mathcal{E}_G$, let $\mathrm{supp}(P)$ be the subgraph of G with edges $\overline{ij}$ for x_{ij} appearing in P. Then we can define a grading on $\mathcal{E}_G$ by letting $\Pi(P)$ be the set partition of $[n]$ that gives the connected components of $\mathrm{supp}(P)$. For example, $\Pi(x_{12}x_{23}x_{45}x_{31}) = 123|45$.