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Relation between the skew-rank of an oriented graph and the rank of its underlying graph

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ABSTRACT

An oriented graph G^σ is a digraph without loops and multiple arcs, where G is called the underlying graph of G^σ. Let $S(G^\sigma)$ denote the skew-adjacency matrix of G^σ, and $A(G)$ be the adjacency matrix of G. The skew-rank of G^σ, written as $sr(G^\sigma)$, refers to the rank of $S(G^\sigma)$, which is always even since $S(G^\sigma)$ is skew symmetric.

A natural problem is: How about the relation between the skew-rank of an oriented graph G^σ and the rank of its underlying graph? In this paper, we focus our attention on this problem. Denote by $d(G)$ the dimension of cycle spaces of G, that is $d(G) = |E(G)| - |V(G)| + \theta(G)$, where $\theta(G)$ denotes the number of connected components of G. It is proved that $sr(G^\sigma) \leq r(G) + 2d(G)$ for an oriented graph G^σ, the oriented graphs G^σ whose skew-rank attains the upper bound are characterized.

1. Introduction

Let G be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. The adjacency matrix $A(G)$ of G is an $n \times n$ symmetric matrix $[a_{xy}]$ such that $a_{xy} = 1$ if x and y are adjacent and $a_{xy} = 0$, otherwise. The rank $r(G)$ of G refers to the rank of $A(G)$. Based on G, an oriented graph G^σ obtained from G is defined by assigning to each edge of G a direction, where G is called the underlying graph of G^σ. The skew-adjacency matrix associated to G^σ, written as $S(G^\sigma)$, is defined to be an $n \times n$ matrix $[s_{xy}]$ such that $s_{xy} = 1$ if there is an arc from x to y, $s_{xy} = -1$ if there is an arc from y to x and $s_{xy} = 0$ otherwise. The skew-rank of G^σ, denoted by $sr(G^\sigma)$, is defined to be the rank of $S(G^\sigma)$, which is even since $S(G^\sigma)$ is skew symmetric.

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Let $C_k = u_1u_2 \cdots u_ku_1$ be a cycle of size k. The sign of C_k^σ with respect to σ is defined to be the sign of $(\Pi_{i=1}^{k-1} s_{u_iu_{i+1}})s_{u_ku_1}$. An even oriented cycle is called evenly-oriented (resp., oddly-oriented) if its sign is positive (resp., negative). An induced subgraph H^σ of G^σ is an oriented graph such that H is an induced subgraph of G and each edge of H^σ has the same orientation as that in G^σ. For $W \subseteq V(G^\sigma)$, $G^\sigma - W$ denotes the subgraph obtained from G^σ by deleting all vertices in W and all incident edges. Sometimes we use the notation $G^\sigma - H^\sigma$ instead of $G^\sigma - V(H^\sigma)$ if H^σ is an induced subgraph of G^σ. For an induced subgraph H^σ and a vertex x outside H^σ, the induced subgraph of G^σ with vertex set $V(H) \cup \{x\}$ is simply written as $H^\sigma + x$. A vertex of G^σ is called a pendant vertex if it is adjacent to a unique vertex, and the unique neighbor of a pendant vertex is called a quasi-pendant vertex. Denote by $d(G)$ the dimension of cycle spaces of G, that is $d(G) = |E(G)| - |V(G)| + \theta(G)$, where $\theta(G)$ denotes the number of connected components of G. Denote by P_n, C_n a path and a cycle of order n, respectively.

Recently, study on skew-adjacency matrices of oriented graphs attracted some attentions. Cavers et al. [5] considered the following topics: graphs whose skew-adjacency matrices are all cospectral; relations between the matching polynomial of a graph and the characteristic polynomial of its adjacency and skew-adjacency matrices; skew-spectral radii and an analogue of Perron–Frobenius theorem; and the number of skew-adjacency matrices of a graph with distinct spectra. Anuradha and Balakrishnan [2] investigated skew spectrum of the Cartesian product of an oriented graph with an oriented Hypercube. Anuradha et al. [3] considered the skew spectrum of special bipartite graphs and solved a conjecture of Cui and Hou [10]. Hou et al. [15] gave an expression of the coefficients of the characteristic polynomial of skew-adjacency matrix $S(G^\sigma)$, and they present new combinatorial proofs for some known results. Gong et al. [13] investigated the coefficients of weighted oriented graphs, and they established recurrences for the characteristic polynomial and deduced a formula for the matching polynomial of an arbitrary weighted oriented graph. Xu [25] established a relation between the spectral radii and the skew spectral radius, some results on the skew spectral radius of an oriented graph and its oriented subgraphs were derived, and a sharp upper bound of the skew spectral radius of oriented unicyclic graphs was present in this paper. For other papers investigating the skew-energy of oriented graphs, we refer the reader to [1,8,12,14,16,17,21–24,26–28].

The rank of undirected graphs have been studied intensively. Cheng et al. [8] characterized undirected graphs with rank 2 or 3; Chang et al. [6,7] respectively determined undirected graphs with rank 4 or 5. In the present paper, we focus our attention on skew-rank of oriented graphs. Li and Yu [18] introduced some preliminary results about the skew-rank of an oriented graph. In [18], the oriented graphs with skew-rank 2 and the oriented graphs with pendant vertices which attain skew-rank 4 were characterized, the skew-rank of oriented unicyclic graphs of order n with girth k in terms of matching number was determined, oriented unicyclic graphs whose skew-adjacency matrices are nonsingular were also described. In [20], the bicyclic oriented graphs with skew-rank 2 or 4 were characterized by Qu and Yu.

For relation between the skew-energy of an oriented graph and the energy of its underlying graph, Adiga et al. [1] proposed the following open problem: Find new families of oriented graphs G^σ whose skew-energy equals the energy of its underlying graph. This problem was partly solved by Hou and Lei in [15]. It is shown in [18] that the skew-rank of an oriented acyclic graph equals the rank of its underlying graph, however the equality often fails to hold for an oriented unicyclic graph or an oriented bicyclic graph. Now a natural problem arises: How about the relation between the skew-rank of an oriented graph G^σ and the rank of its underlying graph? In this paper, we focus our attention on this problem, obtaining the following result.

Theorem 1.1. *Let G^σ be a finite oriented graph without loops and multiple arcs. Then*

$$sr(G^\sigma) \leq r(G) + 2d(G).$$

For purpose of characterizing the extremal graphs whose skew-rank attains the upper bound, an operation on graphs is needed (consulting [18]).

Definition 1.2. Let G be a graph with at least one pendant vertex. The operation of deleting a pendant vertex and its adjacent vertex from G is called δ-transformation.