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Heaps and two exponential structures

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ABSTRACT

Take $\mathsf{Q} = (\mathsf{Q}_1, \mathsf{Q}_2, \ldots)$ to be an exponential structure and $M(n)$ to be the number of minimal elements of Q_n where $M(0) = 1$. Then a sequence of numbers $\{r_n(\mathsf{Q}_n)\}_{n\geq 1}$ is defined by the equation

$$\sum_{n\geq 1} r_n(\mathsf{Q}_n)\frac{z^n}{n!\,M(n)} = -\log\left(\sum_{n\geq 0}(-1)^n\frac{z^n}{n!\,M(n)}\right).$$

Let $\bar{\mathsf{Q}}_n$ denote the poset Q_n with a $\hat{0}$ adjoined and let $\hat{1}$ denote the unique maximal element in the poset Q_n. Furthermore, let μ_{Q_n} be the Möbius function on the poset $\bar{\mathsf{Q}}_n$. Stanley proved that $r_n(\mathsf{Q}_n) = (-1)^n\mu_{\mathsf{Q}_n}(\hat{0}, \hat{1})$. This implies that the numbers $r_n(\mathsf{Q}_n)$ are integers. In this paper, we study the cases $\mathsf{Q}_n = \Pi_n^{(r)}$ and $\mathsf{Q}_n = \mathsf{Q}_n^{(r)}$ where $\Pi_n^{(r)}$ and $\mathsf{Q}_n^{(r)}$ are posets, respectively, of set partitions of $[rn]$ whose block sizes are divisible by r and of r-partitions of $[n]$. In both cases we prove that $r_n(\Pi_n^{(r)})$ and $r_n(\mathsf{Q}_n^{(r)})$ enumerate the pyramids by applying the Cartier–Foata monoid identity and further prove that $r_n(\Pi_n^{(r)})$ is the generalized Euler number E_{rn-1} and that $r_n(\mathsf{Q}_n^{(2)})$ is the number of complete non-ambiguous trees of size $2n-1$ by bijections. This gives a new proof of Welker's theorem that $r_n(\Pi_n^{(r)}) = E_{rn-1}$ and implies the construction of r-dimensional complete non-ambiguous trees. As a bonus of applying the theory of heaps, we establish a bijection between the set of complete non-ambiguous forests and the set of pairs of permutations with no common rise. This answers an open question raised by Aval et al.

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1. Introduction

We denote by Π_n the poset of all the set partitions of $[n]$ ordered by refinement, that is, define $\sigma \leq \pi$ if every block of σ is contained in a block of π. Let $\rho \in \Pi_n$ be the minimal element of Π_n, i.e., $\rho = \{\{1\}, \{2\}, \ldots, \{n\}\}$. Consider an interval $[\sigma, \pi]$ in the poset Π_n and suppose $\pi = \{B_1, B_2, \ldots, B_k\}$ and B_i is partitioned into λ_i blocks in σ. Then we have $[\sigma, \pi] \cong \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}$. For the particular case $\sigma = \rho$, we have $[\rho, \pi] \cong \Pi_{|B_1|} \times \Pi_{|B_2|} \times \cdots \times \Pi_{|B_k|}$. If we set $a_j = |\{i : \lambda_i = j\}|$ for every j, then we can rewrite

$$[\sigma, \pi] \cong \Pi_1^{a_1} \times \Pi_2^{a_2} \times \cdots \times \Pi_n^{a_n}. \tag{1.1}$$

The poset Π_n of set partitions is the archetype of exponential structures. The concept of exponential structure was introduced by Stanley as a generalization of compositional and exponential formulas; see [6,7,3]. An *exponential structure* is a sequence $\mathrm{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \ldots)$ of posets such that:

(1) for each $n \in \mathbb{N}^+$, the poset Q_n is finite, has a unique maximal element $\hat{1}$ and every maximal chain of Q_n has n elements.
(2) for $\pi \in \mathrm{Q}_n$, the interval $[\pi, \hat{1}]$ is isomorphic to the poset Π_k of set partitions for some k.
(3) the subposet $\Lambda_\pi = \{\sigma \in \mathrm{Q}_n : \sigma \leq \pi\}$ of Q_n is isomorphic to $\mathrm{Q}_1^{a_1} \times \mathrm{Q}_2^{a_2} \times \cdots \times \mathrm{Q}_n^{a_n}$ for unique $a_1, a_2, \ldots, a_n \in \mathbb{N}$.

Suppose $\pi \in \mathrm{Q}_n$ and ρ is a minimal element of Q_n satisfying $\rho \leq \pi$. By (1) and (2), we obtain that $[\rho, \hat{1}] \cong \Pi_n$. It follows from (1.1) that $[\rho, \pi] \cong \Pi_1^{a_1} \times \Pi_2^{a_2} \times \cdots \times \Pi_n^{a_n}$ for unique $a_1, a_2, \ldots, a_n \in \mathbb{N}$ satisfying $\sum_i ia_i = n$ and $\sum_i a_i = |\pi|$. In particular, if ρ_1 is another minimal element of Q_n satisfying $\rho_1 \leq \pi$, then we have $[\rho_1, \pi] \cong [\rho, \pi]$.

We will define the numbers $r_n(\mathrm{Q}_n)$ associated with an exponential structure $\mathrm{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \ldots)$ in the following way. Let $M(n)$ be the number of minimal elements of Q_n for $n \geq 1$ and set $M(0) = 1$. Then a sequence of numbers $\{r_n(\mathrm{Q}_n)\}_{n\geq 1}$ is defined by the equation

$$\sum_{n\geq 1} r_n(\mathrm{Q}_n) \frac{z^n}{n!\, M(n)} = -\log\left(\sum_{n\geq 0} (-1)^n \frac{z^n}{n!\, M(n)}\right). \tag{1.2}$$

Let $\bar{\mathrm{Q}}_n$ denote the poset Q_n with a $\hat{0}$ adjoined and let $\hat{1}$ denote the unique maximal element in the poset Q_n. Furthermore, let μ_{Q_n} be the Möbius function on the poset $\bar{\mathrm{Q}}_n$. Then from Chapter 5.5 of [7], we know

$$\sum_{n\geq 1} \mu_{\mathrm{Q}_n}(\hat{0}, \hat{1}) \frac{z^n}{n!M(n)} = -\log\left(\sum_{n\geq 0} \frac{z^n}{n!M(n)}\right), \tag{1.3}$$

and thus, $r_n(\mathrm{Q}_n) = (-1)^n \mu_{\mathrm{Q}_n}(\hat{0}, \hat{1})$. This implies that the numbers $r_n(\mathrm{Q}_n)$ are integers for any exponential structure $\mathrm{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \ldots)$. In the case $\mathrm{Q}_n = \Pi_n$, the number $M(n)$ of minimal elements in the poset Π_n is 1. It follows immediately from (1.3) and (1.2) that $r_n(\Pi_n) = \mu_{\Pi_n}(\hat{0}, \hat{1}) = 0$ for $n \geq 2$ and $r_1(\Pi_1) = -\mu_{\Pi_1}(\hat{0}, \hat{1}) = 1$. There are three other examples of exponential structures $\mathrm{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \ldots)$ in [6,7].

(1) $\mathrm{Q}_n = \mathrm{Q}_n(q)$ which is the poset of direct sum decompositions of the n-dimensional vector space $V_n(q)$ over the finite field $\mathbb{F}_q$. Let $V_n(q)$ be an n-dimensional vector space over the finite field $\mathbb{F}_q$. Let $\mathrm{Q}_n(q)$ consist of all the collections $\{W_1, W_2, \ldots, W_k\}$ of subspaces of $V_n(q)$ such that $\dim W_i > 0$ for all i, and such that $V_n(q) = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ (direct sum). An element of $\mathrm{Q}_n(q)$ is called a *direct sum decomposition* of $V_n(q)$. We order $\mathrm{Q}_n(q)$ by refinement, i.e., $\{W_1, W_2, \ldots, W_k\} \leq \{W'_1, W'_2, \ldots, W'_j\}$ if each W_r is contained in some W'_s.
(2) $\mathrm{Q}_n = \Pi_n^{(r)}$ which is the poset of set partitions of $[rn]$ whose block sizes are divisible by r.
(3) $\mathrm{Q}_n = \mathrm{Q}_n^{(r)}$ which is the poset of r-partitions of $[n]$. The definition of r-partition will be given in Section 4.