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On a generalization of a theorem of Sárközy and Sós

Yong-Gao Chen [a], Min Tang [b]

[a] School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, PR China

[b] School of Mathematics and Computer Science, Anhui Normal University, Wuhu 241003, PR China

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ABSTRACT

Let $\mathbb{N}_0$ be the set of all nonnegative integers and $\ell \geq 2$ be a fixed integer. For $A \subseteq \mathbb{N}_0$ and $n \in \mathbb{N}_0$, let $r'_\ell(A, n)$ denote the number of solutions of $a_1 + \cdots + a_\ell = n$ with $a_1, \ldots, a_\ell \in A$ and $a_1 \leq \cdots \leq a_\ell$. Let k be a fixed positive integer. In this paper, we prove that, for any given distinct positive integers u_i $(1 \leq i \leq k)$ and positive rational numbers α_i $(1 \leq i \leq k)$ with $\alpha_1 + \cdots + \alpha_k = 1$, there are infinitely many sets $A \subseteq \mathbb{N}_0$ such that $r'_\ell(A, n) \geq 1$ for all $n \geq 0$ and the set of n with $r'_\ell(A, n) = u_i$ has density α_i for all $1 \leq i \leq k$.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0$ be the set of all nonnegative integers. Let $\ell \geq 2$ be a fixed integer. For $A \subseteq \mathbb{N}_0$, $n \in \mathbb{N}_0$, and $N \in \mathbb{N}$, let

$$r_\ell(A, n) = \sharp\{(a_1, a_2, \ldots, a_\ell) \in A^\ell : a_1 + a_2 + \cdots + a_\ell = n\},$$
$$r'_\ell(A, n) = \sharp\{(a_1, a_2, \ldots, a_\ell) \in A^\ell : a_1 + a_2 + \cdots + a_\ell = n,\ a_1 \leq a_2 \leq \cdots \leq a_\ell\},$$
$$\mathcal{S}_u^{(\ell)}(A) = \{n \in \mathbb{N} : r'_\ell(A, n) = u\},$$
$$\mathcal{S}_u^{(\ell)}(A, N) = \sharp\{n \leq N : r'_\ell(A, n) = u\}.$$

The subset A of $\mathbb{N}_0$ is called *a basis of order* ℓ if $r'_\ell(A, n) \geq 1$ for all $n \geq 0$.

The well-known Erdős–Turán conjecture [3] asserts that if A is a basis of order 2, then $r_2(A, n)$ is unbounded. It is also well known by now that the counterpart of the Erdős–Turán conjecture does

E-mail addresses: ygchen@njnu.edu.cn (Y.-G. Chen), tmzzz2000@163.com (M. Tang).

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not hold in many families of semigroups. Unfortunately, this conjecture itself is still a major unsolved problem in additive number theory. Several mathematicians improved the known lower bound of $\limsup_{n\to\infty} r_2(A,n)$ for all bases A. In 2003, Grekos et al. [4] proved that if A is a basis of order 2, then $\limsup_{n\to\infty} r_2(A,n) \ge 6$. In 2005, Borwein et al. [1] improved 6 to 8. In 2013, Konstantoulas [5] proved that, if the upper density of the set of numbers not represented as sums of two elements of A is less than $1/10$, then $\limsup_{n\to\infty} r_2(A,n) \ge 6$.

In 2012, the first author of this paper [2] proved that there exists a basis A of order 2 of $\mathbb{N}$ such that the set of n with $r_2(A,n) = 2$ has density one. In 2013, Yang [8] generalized Chen's method to prove that for any integer $k \ge 2$, there exists a basis A of order k such that the set of n with $r_k(A,n) = k!$ has density one. The second author of this paper [7] developed Chen and Yang's method of proof to establish the following more general result: For any fixed integers $k \ge 2$ and $u \ge 1$, there exists a basis A of order k such that $r_k(A,n) \ge 1$ for all $n \ge 0$ and the set of n with $r_k(A,n) = k!u$ has density one. In 1997, Sárközy and Sós [6] considered a similar problem and they showed that for every finite set $U \in \mathbb{N}$ there is a set A such that, apart from a "thin" set of integers n, $r_2'(A,n)$ assumes only the prescribed values $u \in \mathbb{U}$ with about the same frequency. In detail, they proved the following result.

Theorem A. *Let $k \in \mathbb{N}$ and let $u_1 < u_2 < \cdots < u_k$ be positive integers. Then there is an infinite set $A \subset \mathbb{N}_0$ such that writing*

$$B = \mathbb{N} \setminus (\cup_{i=1}^{k} \mathcal{S}_{u_i}^{(2)}(A))$$

we have

$$\mathcal{S}_{u_i}^{(2)}(A,N) = \frac{N}{k} + O(N^{\alpha})$$

and

$$B(N) = O(N^{\alpha})$$

where $\alpha = \log 3/\log 4$ and $B(N) = |B \cap [1,N]|$.

Let $r_i \in \mathbb{Q}$, $1 \le i \le k$ with $\sum_{i=1}^{k} r_i = 1$. Sárközy and Sós (See [6, Remark 4.1]) remarked that using the same idea as in the proof of Theorem A, they can prove the existence of an infinite set $A \subset \mathbb{N}_0$ for which

$$\mathcal{S}_{u_i}^{(2)}(A,N) = r_i N + O(N^{\alpha}), \quad 1 \le i \le k$$

with some $0 < \alpha < 1$.

In this paper, we extend Sárközy and Sós's result to $\ell \ge 2$. We find that it is difficult to handle the cases $\ell \ge 3$ by using Sárközy and Sós's method. The method used here is different from Sárközy and Sós's method.

Theorem 1. *Let $k, \ell \in \mathbb{N}$ with $\ell \ge 2$ and let $u_1 < u_2 < \cdots < u_k$ be positive integers. Let α_i $(1 \le i \le k)$ be positive rational numbers with $\alpha_1 + \cdots + \alpha_k = 1$. Then there are infinitely many bases A of order ℓ such that*

$$\mathcal{S}_{u_i}^{(\ell)}(A,N) = \alpha_i N + O(N^{\alpha}), \quad 1 \le i \le k, \tag{1}$$

where $\alpha = \alpha(A)$ with $0 < \alpha < 1$.

Let $B = \mathbb{N} \setminus (\cup_{i=1}^{k} \mathcal{S}_{u_i}^{(\ell)}(A))$. If (1) holds, then $B(N) = O(N^{\alpha})$.

2. Proofs

For $X_i \subseteq \mathbb{Z}$ $(1 \le i \le t)$, let

$$X_1 + \cdots + X_t = \{x_1 + \cdots + x_t : x_i \in X_i (1 \le i \le t)\}.$$

For $X \subseteq \mathbb{Z}$ and $n \in \mathbb{N}$, let

$$n \times X = \{nx : x \in X\}.$$