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A short proof of the equivalence of left and right convergence for sparse graphs

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ABSTRACT

There are several notions of convergence for sequences of bounded degree graphs. One such notion is left convergence, which is based on counting neighborhood distributions. Another notion is right convergence, based on counting homomorphisms to a (weighted) target graph. Borgs, Chayes, Kahn and Lovász showed that a sequence of bounded degree graphs is left convergent if and only if it is right convergent for certain target graphs H with all weights (including loops) close to 1. We give a short alternative proof of this statement. In particular, for each bounded degree graph G we associate functions $f_{G,k}$ for every positive integer k, and we show that left convergence of a sequence of graphs is equivalent to the convergence of the partial derivatives of each of these functions at the origin, while right convergence is equivalent to pointwise convergence. Using the bound on the maximum degree of the graphs, we can uniformly bound the partial derivatives at the origin, and show that the Taylor series converges uniformly on a domain independent of the graph, which implies the equivalence.

1. Introduction

The theory of graph limits has been extensively developing in recent years, primarily in the dense case [7,8]. For sequences of sparse graphs, in particular, sequences of graphs with bounded degree, much less is known. A notion of convergence, which we will refer to as left convergence, was first defined by Benjamini and Schramm [3]. One of the major open problems on bounded degree graph

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sequences, the conjecture of Aldous and Lyons [1] on left convergent sequences of graphs, is very closely related to Gromov's question about whether all countable discrete groups are sofic [10]. In a certain sense, left convergence is too rough to "distinguish" graphs which should be different. Because of this, other, finer notions of convergence for sequences of bounded degree graphs [4,5,11] have been proposed. The full picture of their mutual relations is not completely understood. Borgs, Chayes, Kahn and Lovász [6] prove that a sequence of bounded degree graphs is left convergent if and only if it is "right convergent" for a certain set of target graphs. We provide an alternative, shorter proof of this statement, which we believe also gives some new and useful insights.

Here and throughout this paper we use the following setting: $G_n = (V(G_n), E(G_n))$, is a sequence of graphs with degrees bounded uniformly by a constant D, and $v(G_n) = |V(G_n)| \to \infty$. Given two simple graphs G and H, we let $\hom(G, H)$ denote the number of maps $V(G) \to V(H)$ such that any edge of G is mapped to an edge of H. We define $\mathrm{inj}(G, H)$ as the number of maps that are injective on the set of vertices, and $\mathrm{ind}(G, H)$ as the number of injective maps that are isomorphisms between G and the subgraph of H induced by the images of the vertices in G.

Suppose now that H is a weighted graph, with vertex weights w_i and edge weights w_{ij}. We think of each pair of vertices as having a weight, perhaps equal to 0. We define $\hom(G, H)$ as the sum over all maps $f : V(G) \to V(H)$ of the product

$$\prod_{v\in G} w_{f(v)} \prod_{(uv)\in E(G)} w_{f(u)f(v)}.$$

Note that if all vertices in H have weight 1 and all edges have weight 0 or 1, then $\hom(G, H)$ is equal to $\hom(G, H')$ where H' is the unweighted graph on the vertex set of H formed by the edges with weight 1. We also define $t(G, H)$ as the *average* of this product over all maps, this simply divides it by a factor of $v(H)^{v(G)}$.

One important notion of convergence is the following, defined by Benjamini and Schramm [3]:

Definition 1 (*Left convergence*)**.** Given a sequence of graphs (G_n) with all degrees bounded by a positive integer D, we say that the sequence is left convergent if for any connected graph F, the limit

$$\lim_{n\to\infty} \frac{\mathrm{ind}(F, G_n)}{v(G_n)}$$

exists. It is well known that we obtain an equivalent definition if we replace *ind* with *inj* or *hom*.

Note that the original definition by Benjamini and Schramm involved looking at the distribution of the r-neighborhoods of vertices for any r, but it is easy to see that the two notions are equivalent.

We also examine the notion of right convergence:

Definition 2 (*Right convergence*)**.** Given a sequence of graphs (G_n) with bounded degrees, we say that the sequence is right convergent with soft-core constraints, or simply soft-core right convergent, if for any weighted target graph H that is complete and has positive weights (including on loops), the limit

$$\lim_{n\to\infty} \frac{\log \hom(G_n, H)}{v(G_n)}$$

exists. If the limit also exists for all graphs H with nonnegative edge weights, we will refer to it as right convergence with hard-core constraints, or hard-core right convergence.

If we consider $t(G_n, H)$ instead of $\hom(G_n, H)$, we obtain an equivalent definition, this simply decreases each term in the sequence by the same constant, $\log v(H)$. Given a sequence G_n of bounded degree graphs, if for each n we add or delete $o(v(G_n))$ edges in G_n, neither soft-core right convergence, nor left convergence is affected. However, hard-core right convergence can change. Borgs, Chayes, and Gamarnik [5] show that if a sequence is soft-core right convergent, then one can delete $o(v(G_n))$ edges from G_n to make it hard-core right convergent. In particular, this implies that if every hard-core right convergent sequence is left convergent, then every soft-core right convergent sequence is left convergent.