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A matroid analogue of a theorem of Brooks for graphs

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ABSTRACT

Brooks proved that the chromatic number of a loopless connected graph G is at most the maximum degree of G unless G is an odd cycle or a clique. This note proves an analogue of this theorem for $GF(p)$-representable matroids when p is prime, thereby verifying a natural generalization of a conjecture of Peter Nelson.

1. Introduction

The matroid terminology and notation used here will follow [8]. For a matroid M having ground set E and rank function r, the *chromatic* or *characteristic polynomial* of M is defined by

$$p(M;\lambda) = \sum_{X\subseteq E}(-1)^{|X|}\lambda^{r(M)-r(X)}.$$

If M is the cycle matroid of a graph G and G has $\omega(G)$ components, then the chromatic polynomial $P_G(\lambda)$ of the graph G is linked to the chromatic polynomial of its cycle matroid $M(G)$ via the following equation:

$$P_G(\lambda) = \lambda^{\omega(G)}p(M(G);\lambda).$$

Of course, the chromatic number $\chi(G)$ of G is the smallest positive integer j for which $P_G(j)$ is positive unless G has a loop, in which case, the chromatic number is ∞. Let M be a rank-r simple matroid that is representable over $GF(q)$ and let T be a subset of $PG(r-1,q)$ such that $M \cong PG(r-1,q)|T$. Let Q be a flat of $PG(r-1,q)$ that avoids T and has maximum rank. The *critical exponent* $c(M;q)$ of M is $r - r(Q)$. If M is loopless but has parallel elements, we define $c(M;q) = c(\mathrm{si}(M);q)$. If M has a loop,

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$c(M; q) = \infty$. Ostensibly, $c(M; q)$ depends on the embedding of M in $PG(r-1, q)$ but the following fundamental result of Crapo and Rota [4] establishes that this is not the case.

Theorem 1.1. *Let M be a loopless matroid that is representable over GF(q). Then*

$$c(M; q) = \min\{j : p(M; q^j) > 0\}.$$

Evidently, the critical exponent is an analogue of the chromatic number of a graph. Indeed, Geelen and Nelson [5] use the term 'critical number' rather than 'critical exponent' to highlight this analogy. For a loopless graph G, it is immediate that, for all prime powers q,

$$q^{c(M(G);q)-1} < \chi(G) \leq q^{c(M(G);q)}.$$

Brooks [1] proved the following well-known result. For a graph G, let $\Delta(G)$ denote its maximum vertex degree.

Theorem 1.2. *Let G be a loopless connected graph. Then*

$$\chi(G) \leq \Delta(G) + 1.$$

Indeed, $\chi(G) \leq \Delta(G)$ *unless G is an odd cycle or a complete graph.*

The purpose of this note is to prove the following analogue of this result for $GF(q)$-representable matroids when q is prime. This new result was essentially conjectured by Peter Nelson [7]. An alternative analogue of Brooks' Theorem, one for regular matroids, was proved in [9, Theorem 2.12].

Theorem 1.3. *Let p be a prime and M be a loopless non-empty GF(p)-representable matroid whose largest cocircuit has* c^* *elements. Then*

$$c(M; p) \leq \lceil \log_p(1 + c^*) \rceil.$$

Indeed, if M is connected, then $c(M; p) \leq \lceil \log_p c^* \rceil$ *unless M is a projective geometry or M is an odd circuit, where the latter only occurs when* $p = 2$.

The requirement that M be connected appears in the last part of the theorem only to streamline the statement. It is not difficult to state a result in the absence of that requirement since the critical exponent of a loopless matroid M is the maximum of the critical exponents of its components while the maximum cocircuit size of M is the maximum of the maximum cocircuit sizes of its components.

We conjecture that Theorem 1.3 remains true if p is replaced by an arbitrary prime power q, but the proof technique used here only works when q is prime.

2. The proof

The proof of the main result will use three lemmas, the first of which is [9, Theorem 3.5]. For a matroid M, let $\mathcal{R}(M)$ be the set of simple restrictions of M, and let $\mathcal{C}^*(M)$ be the set of cocircuits of M.

Lemma 2.1. *Let M be a GF(q)-representable matroid having no loops. Then*

$$c(M; q) \leq \lceil \log_q(1 + \max_{N \in \mathcal{R}(M)} (\min_{C^* \in \mathcal{C}^*(N)} |C^*|)) \rceil.$$

Murty [6] considered the class of matroids in which all circuits have the same cardinality. His main result, which can be stated as follows, determined all binary matroids with this property.

Lemma 2.2. *Let M be a connected binary matroid with at least two elements. Then every cocircuit of M has the same cardinality if and only if, for some positive integer t, the matroid M can be obtained by adding* $t-1$ *elements in parallel to each element of one of the following:*

(i) $U_{r,r+1}$ *for some* $r \geq 2$*;*
(ii) $PG(r-1, 2)$ *for some* $r \geq 1$*; or*
(iii) $AG(r-1, 2)$ *for some* $r \geq 2$.