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A formula for simplicial tree-numbers of matroid complexes

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ABSTRACT

We give a formula for the simplicial tree-numbers of the independent set complex of a finite matroid M as a product of eigenvalues of the total combinatorial Laplacians on this complex. Two matroid invariants emerge naturally in describing the multiplicities of these eigenvalues in the formula: one is the unsigned reduced Euler characteristic of the independent set complex and the other is the β-invariant of a matroid. We will demonstrate various applications of this formula including a "matroid theoretic" derivation of Kalai's simplicial tree-numbers of a standard simplex.

1. Introduction

In this paper, we will prove a formula for the simplicial tree-numbers of the independent set complex $IN(M)$ of a finite matroid M. Simplicial trees for simplicial complexes have been studied as a generalization of spanning trees for graphs [7,1,4]. In these studies, combinatorial Laplacians for simplicial complexes play a role analogous to that of graph Laplacians for graphs. We will review simplicial tree-numbers and combinatorial Laplacians in Section 2. Since eigenvalues of combinatorial Laplacians on matroid complexes are known [11], one may ask whether the simplicial tree-numbers for these complexes can be computed [12,4]. We give an answer to this question in this paper.

The formula is given as a product of eigenvalues of the combinatorial Laplacians on $IN(M)$. We refer the readers to [11] for the integrality of these eigenvalues. Two matroid invariants will appear naturally in describing the multiplicities of the eigenvalues in this product. One is the unsigned reduced Euler characteristic of $IN(M)$, and the other is Crapo's β-invariant of M. For the sake of

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simplicity, we will refer to the former as the α-invariant of M, and denote the two invariants by $\alpha(M)$ and $\beta(M)$, respectively. We will review these invariants in Section 2, also.

Our main result (Theorem 5) can be stated as follows. Let M be a matroid of rank $d+1$ on a finite ground set E, and $L(M)$ its lattice of flats. For $V \in L(M)$, let M/V be the contraction of V from M. For an integer $\lambda \in [0, |E|]$, we define $L(M)_\lambda = \{V \in L(M) : |E \setminus V| = \lambda\}$, and define the *convolution* $\alpha \circ_\lambda \beta$ of α-invariant and β-invariant for M with respect to the given λ to be

$$\alpha \circ_\lambda \beta = \sum_{V \in L(M)_\lambda} \alpha(V)\beta(M/V).$$

Then, the simplicial tree-number k_d of $IN(M)$ in the top dimension d is

$$k_d = \prod_{\lambda \in (0,|E|]} \lambda^{\alpha \circ_\lambda \beta}$$

where the product is over all positive integers $\lambda \in (0, |E|]$ (with $L(M)_\lambda \neq \emptyset$).

This formula can be used to give simplicial tree-numbers k_i for $IN(M)$ in other dimensions i. For $-1 \leq i \leq d$, we observe that k_i is the top dimensional simplicial tree-number of $IN(T^{i+1}(M))$, where $T^{i+1}(M)$ is the *truncation* obtained from M by ignoring all independent sets of rank $> i+1$. Therefore, the main result can be applied to the matroid $T^{i+1}(M)$ to compute k_i. We shall demonstrate these computations via an example later.

This paper is organized as follows. In Section 2, we will review all preliminary definitions and results concerning simplicial tree-numbers, combinatorial Laplacians, and $\alpha(M)$ and $\beta(M)$ of a finite matroid M. In Section 3, we will present a proof of the main result, Theorem 5. Section 4 will discuss applications of Theorem 5, including a new derivation of Kalai's classical result on standard simplex [7]. We refer the readers to [13] or [15] for terminologies and definitions from matroid theory that are used in this paper.

2. Preliminaries

2.1. Simplicial tree-numbers

In this paper, *simplicial tree-numbers* mean high-dimensional tree-numbers of a simplicial complex (refer to [4,5,8]). A non-empty simplicial complex Γ is said to be $\mathbb{Z}$-*acyclic in positive codimensions* (shortly, $\mathbb{Z}$-*APC*) if the reduced integral homology $\tilde{H}_i(\Gamma) = 0$ for $i < \dim \Gamma$ (refer to [4]). Note that the independent set complex $IN(M)$ of a finite (non-empty) matroid M is $\mathbb{Z}$-*APC* because it is shellable (refer to [2]).

Let Γ be a $\mathbb{Z}$-APC complex of dimension d. For $i \in [0, d]$, let Γ_i denote the set of all i-dim simplices in Γ. The i-skeleton of Γ is $\Gamma^{(i)} = \Gamma_{-1} \cup \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_i$, where we define $\Gamma_{-1} = \{\emptyset\}$. For a non-empty subset $S \subset \Gamma_i$, define $\Gamma_S := S \cup \Gamma^{(i-1)}$ as an i-dimensional subcomplex of Γ. For $i \in [-1, d]$, a non-empty subset $B \subset \Gamma_i$ is an *i-dimensional simplicial tree* (or simply, *i-tree*) if

(1) $\tilde{H}_i(\Gamma_B) = 0$, and
(2) $|\tilde{H}_{i-1}(\Gamma_B)|$ is finite.

Note that $\tilde{H}_j(\Gamma_B) = 0$ for $j \leq i-2$ is a consequence of the fact $\Gamma_B^{(i-1)} = \Gamma^{(i-1)}$. We will denote the set of all i-trees in Γ by $\mathcal{B}_i = \mathcal{B}_i(\Gamma)$ with $\mathcal{B}_{-1} = \{\{\emptyset\}\}$.

Define the i-th simplicial tree-number (or simply, i-th *tree-number*) of Γ to be

$$k_i = k_i(\Gamma) = \sum_{B \in \mathcal{B}_i} |\tilde{H}_{i-1}(\Gamma_B)|^2.$$

We have $k_{-1} = 1$ by definition, and $k_0 = |\Gamma_0|$. If Γ is a connected graph, then k_1 is the number of spanning trees in Γ because $|\tilde{H}_0(\Gamma_B)| = 1$ for $B \in \mathcal{B}_1$. However, $|\tilde{H}_{i-1}(\Gamma_B)|$ may not equal 1 for $B \in \mathcal{B}_i$ when $i > 1$. Refer to [7] for an example.