European Journal of Combinatorics 53 (2016) 66–71

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Domination number and Laplacian eigenvalue distribution

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ARTICLE INFO

Article history:
Received 7 April 2015
Accepted 18 November 2015
Available online 7 December 2015

ABSTRACT

Let $m_G(I)$ denote the number of Laplacian eigenvalues of a graph G in an interval I. Our main result is that for graphs having domination number γ, $m_G[0, 1) \leq \gamma$, improving existing bounds in the literature. For many graphs, $m_G[0, 1) = \gamma$, or $m_G[0, 1) = \gamma - 1$.

1. Introduction

Given $G = (V, E)$, an undirected graph with vertex set $V = \{v_1, \ldots, v_n\}$, its *adjacency matrix* $A = [a_{ij}]$ is the $n \times n$ 0–1 matrix for which $a_{ij} = 1$ if and only if v_i and v_j are adjacent. The *Laplacian matrix* of G is defined as $L_G = D - A$, where $D = [d_{ij}]$ is the diagonal matrix in which $d_{ii} = \deg(v_i)$, the degree of v_i. The *Laplacian spectrum* of G is the multi-set of eigenvalues of L_G, we number $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. We write $\lambda_i(G)$ for λ_i if ambiguity exists. All eigenvalues in this paper are Laplacian, and we refer to [19,20] for more background. Throughout, δ denotes the minimum vertex degree.

It is known that for graphs G of order n, $\lambda_1 \leq n$ with equality if and only if $\bar{G}$ is disconnected ([17], p. 148). For trees T, $\lambda_1 = n$ if and only if T is the star S_n. We are interested in the distribution of Laplacian eigenvalues of graphs. Given a real interval I, and graph G, we let $m_G(I)$ denote the number of Laplacian eigenvalues, multiplicities included, of G in I.

In [16] Merris gives a lower bound for $m_G(2, n]$ in connected graphs of order $n > 2$, and [10] contains many results including lower bounds for $m_G[0, 1)$ and $m_G[1, n]$. Recall that the *distance* between two vertices u and v is the number of edges in a shortest path between them, and the graph's

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http://dx.doi.org/10.1016/j.ejc.2015.11.005
0195-6698/

diameter is the greatest distance between any two vertices. For trees T it is known (see [10]) that $m_T(0, 2) \geq \lfloor \frac{\text{diam}}{2} \rfloor \leq m_T(2, n)$. Guo [11] showed that in trees, $\lambda_k \leq \lceil \frac{n}{k} \rceil$ for each k, $1 \leq k \leq n$.

A graph's *edge covering number*, denoted α_1, is the minimum number of edges needed to cover all vertices. Its *matching number* β_1 is the maximum number of independent edges. In a graph without isolates, they are related by the Gallai formula $\alpha_1 + \beta_1 = n$. In [12] Guo et al. showed (Thr. 4 and Cor. 6) that for connected graphs G, $m_G[1, n] \geq \alpha_1$ and $m_G[0, 1) \leq \beta_1$.

A set $S \subseteq V$ is *dominating* if every $v \in V - S$ is adjacent to some member in S. The *domination number* γ is the minimum size of a dominating set. The corresponding decision problem for γ is known to be NP-complete, even for planar graphs [9].

There is a considerable body of results that relate γ and the Laplacian spectrum. The earliest such work seems to be the 1996 paper [5] in which Brand and Seifter showed that $\lambda_1 < n - \lceil \frac{\gamma-2}{2} \rceil$, for connected graphs with $\gamma \geq 3$. Recently in [25], their upper bound for λ_1, called the *Laplacian spectral radius*, was improved by Xing and Zhou who showed $\lambda_1 \leq n - \gamma + 2$, when $2 \leq \gamma \leq n - 1$. In [21] Nikiforov obtained the lower bound $\lambda_1 \geq \lceil \frac{n}{\gamma} \rceil$ when $n \geq 2$, and characterized when equality occurs.

Bounds for the second smallest Laplacian eigenvalue λ_{n-1}, known as the *algebraic connectivity*, can be found in [1] as well as [13], where it is shown that if G has no isolated vertices, then $\lambda_{n-1} \leq n - 2(\gamma - 1)$.

Studies relating the domination number to the Laplacian spectrum of trees include [8], in which the authors determined which trees attain minimal Laplacian spectral radius, where $n = k\gamma$ and $\gamma = 2, 3, 4$. In [26] it is shown that $m_T[0, 2) \leq n - \gamma$ in trees T. The domination number also appears in spectral studies of the adjacency, signless Laplacian, and distance matrices, including [25] and papers cited therein.

Our main result is that if G has domination number γ, then $m_G[0, 1) \leq \gamma$ and $m_G[1, n] \geq n - \gamma$. Since it is known [14, p. 58–59] that in isolate-free graphs, $\gamma \leq \beta_1$, and so $n - \gamma \geq n - \beta_1 = \alpha_1$, our results provide improvements over the bounds in [12]. Observe that we cannot say $m_T(1, n] \geq n - \gamma$, since the star S_3 has domination number 1, and Laplacian spectrum 0, 1, 3.

In the next section, we give some necessary lemmas. In Section 3 we give several upper bounds on γ, and prove our main result. In Section 4 we observe that for many graphs, $m_G[0, 1)$ and γ are equal or close. We also give formulas for $m_G[0, 1)$ when G is a path or cycle, and exhibit a tree T for which $m_T[0, 1) < \gamma$. In Section 5 we obtain some lower bounds. For graphs having a hamiltonian path or cycle there are at least $2\lfloor \frac{n}{3} \rfloor$ eigenvalues in $[1, n]$, and at least $\lfloor \frac{n}{3} \rfloor$ in $[3, n]$. Using a theorem of Egawa and Ota we show that if G has order $n \geq 10$, with $\delta \geq \frac{n+2}{4}$, then nearly three-fourths of its Laplacian eigenvalues are in $[1, n]$, and nearly one-fourth are in $[4, n]$.

2. Preliminaries

In this paper, a *star* S_n is the complete bipartite graph $K_{1,n-1}$, and $n \geq 2$. Lemma 1 is well-known and Lemma 2 is easy to show. Lemma 3 is the Weyl monotonicity theorem, also well-known (e.g. [6, Thr. 2.8.1]).

Lemma 1. *The star S_n on n vertices has Laplacian spectrum* $0, 1^{n-2}, n$.

Lemma 2. *For graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ where $E_1 \cap E_2 = \emptyset$, and $G = (V, E_1 \cup E_2)$, we have $L_G = L_{G_1} + L_{G_2}$.*

Lemma 3. *If A and B are Hermitian matrices of order n, and B is positive semi-definite, then $\lambda_i(A + B) \geq \lambda_i(A)$, for $1 \leq i \leq n$.*

Lemma 4. *Let $G = (V, E)$ and $H = (V, F)$ be graphs with $F \subseteq E$. Then*

(1) *for all i, $\lambda_i(H) \leq \lambda_i(G)$;*
(2) *for any a, $m_H[0, a) \geq m_G[0, a)$;*
(3) *for any a, $m_H[a, n] \leq m_G[a, n]$.*