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Positive graphs

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ABSTRACT

We study "positive" graphs that have a nonnegative homomorphism number into every edge-weighted graph (where the edgeweights may be negative). We conjecture that all positive graphs can be obtained by taking two copies of an arbitrary simple graph and gluing them together along an independent set of nodes. We prove the conjecture for various classes of graphs including all trees. We prove a number of properties of positive graphs, including the fact that they have a homomorphic image which has at least half the original number of nodes but in which every edge has an even number of pre-images. The results, combined with a computer program, imply that the conjecture is true for all but one graph up to 10 nodes.

1. Problem description

For a graph G we are going to denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$, but may simply write V and E when it is clear from the context which graph we are talking about.

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Let G and H be two simple graphs. A *homomorphism* $G \to H$ is a map $V(G) \to V(H)$ that preserves adjacency. We denote by $\hom(G, H)$ the number of homomorphisms $G \to H$. We extend this definition to graphs H whose edges are weighted by real numbers $\beta_{ij} = \beta_{ji}$ $(i, j \in V(H))$:

$$\hom(G, H) = \sum_{f:\, V(G)\to V(H)} \prod_{ij\in E(G)} \beta_{f(i)f(j)}.$$

(One could extend it further by allowing nodeweights, and also by allowing weights in G. Positive nodeweights in H would not give anything new; whether we get anything interesting through weighting G is not investigated in this paper.)

We call the graph G *positive* if $\hom(G, H) \geq 0$ for every edge-weighted graph H (where the edgeweights may be negative). It would be interesting to characterize these graphs; in this paper we offer a conjecture and line up supporting evidence.

We call a graph *symmetric*, if its vertices can be partitioned into three sets (S, A, B) so that S is an independent set, there is no edge between A and B, and there exists an isomorphism between the subgraphs spanned by $S \cup A$ and $S \cup B$ which fixes S.

Conjecture 1. *A graph G is positive if and only if it is symmetric.*

There is an analytic definition for graph positivity which is sometimes more convenient to work with. A *kernel* is a symmetric bounded measurable function $W : [0, 1]^2 \to \mathbb{R}$. A map $p : V(G) \to [0, 1]$ can be thought of as a homomorphism into W. It also naturally induces a map $p : E(G) \to [0, 1]^2$. The *weight* of $p \in [0, 1]^{V(G)}$ is then defined as

$$\hom(G, W, p) = \prod_{e\in E} W\big(p(e)\big) = \prod_{(a,b)\in E} W\big(p(a), p(b)\big).$$

The *homomorphism density* of a graph $G = (V, E)$ in a kernel W is defined as the expected weight of a random map:

$$t(G, W) = \int_{[0,1]^V} \hom(G, W, p)\, \mathrm{d}p = \int_{[0,1]^V} \prod_{e\in E} W\big(p(e)\big)\, \mathrm{d}p. \tag{1}$$

Graphs with real edge weights can be considered as kernels in a natural way: Let H be a looped-simple graph with edge weights β_{ij}; assume that $V(H) = [n] = \{1, \ldots, n\}$. Split the interval $[0, 1]$ into n intervals $J_1, \ldots, J_n$ of equal length, and define

$$W_H(x, y) = \beta_{ij} \quad \text{for } x \in J_i,\ y \in J_j.$$

Then it is easy to check that for every simple graph G and edge-weighted graph H, we have $t(G, W_H) = t(G, H)$, where $t(G, H)$ is a normalized version of homomorphism numbers between finite graphs:

$$t(G, H) = \frac{\hom(G, H)}{|V(H)|^{|V(G)|}}.$$

(For two simple graph G and H, $t(G, H)$ is the probability that a random map $V(G) \to V(H)$ is a homomorphism.)

It follows from the theory of graph limits [1,5] that positive graphs can be equivalently be defined by the property that $t(G, W) \geq 0$ for every kernel W.

Hatami [3] studied “norming” graphs G, for which the functional $W \mapsto t(G, W)^{|E(G)|}$ is a norm on the space of kernels. Positivity is clearly a necessary condition for this (it is far from being sufficient, however). We do not know whether our Conjecture can be proved for norming graphs.