European Journal of Combinatorics 52 (2016) 338–367

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A new line of attack on the dichotomy conjecture

Gábor Kun [a,1], Mario Szegedy [b,2]

[a] Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences Reáltanoda utca 13-15, H-1053, Budapest, Hungary

[b] Department of Computer Science Rutgers, the State University of New Jersey, 110 Frelinghuysen Road Piscataway, NJ 08854-8019, USA

ARTICLE INFO

Article history:
Available online 13 August 2015

ABSTRACT

The well known dichotomy conjecture of Feder and Vardi states that for every finite family Γ of constraints CSP(Γ) is either polynomially solvable or *NP*-hard. Bulatov and Jeavons reformulated this conjecture in terms of the properties of the algebra *Pol*(Γ), where the latter is the collection of those n-ary operations ($n = 1, 2, \ldots$) that keep all constraints in Γ invariant. We show that the algebraic condition boils down to whether there are arbitrarily resilient functions in *Pol*(Γ). Equivalently, we can express this in terms of the PCP theory: CSP(Γ) is *NP*-hard iff every long code test created from Γ that passes with zero error admits only juntas.[3] Then, using this characterization and a result of Dinur, Friedgut and Regev, we give an entirely new and transparent proof to the Hell–Nešetřil theorem, which states that for a simple, connected and undirected graph H, the problem CSP(H) is *NP*-hard if and only if H is non-bipartite.

We also introduce another notion of resilience (we call it strong resilience), and we use it in the investigation of CSP problems that 'do not have the ability to count'. We show that CSP problems without the ability to count are exactly the ones with strongly resilient term operations. This gave already a handier tool to attack the conjecture that CSP problems without the ability to count have

E-mail addresses: kungabor@cs.elte.hu (G. Kun), szegedy@cs.rutgers.edu (M. Szegedy).

[1] Tel.: +36 1 483 8302; fax: +36 1 483 8333.

[2] Tel.: +1 (732) 445 2001x4184; fax: +1 (732) 445 0537.

[3] For us "junta" means that a constant number of the variables have constant influence on the outcome. Also, Γ need to be a core (see definition 8).

http://dx.doi.org/10.1016/j.ejc.2015.07.011

bounded width, or equivalently, that they can be characterized by existential k-pebble games: Barto and Kozik already proved this conjecture using a variant of our characterization. This is considered a major step towards the resolution of the dichotomy conjecture.

Finally, we show that a yet stronger notion of resilience, when the term operation is asymptotically constant, holds for the class of *width one* CSPs.

What emerges from our research, is that certain important algebraic conditions that are usually expressed via identities have equivalent analytic definitions that rely on asymptotic properties of term operations.

1. Introduction

Constraint satisfaction problems (CSP) are the pinnacles in *NP* not only because they have multiple interpretations in logic, combinatorics, and complexity theory, but also for their immense popularity in various branches of science and engineering, where they are looked at as a versatile language for phrasing search problems. This said, it is even more remarkable that some basic complexity questions about them remain unanswered.

To a finite domain D, variables $\{x_1, x_2, \ldots\}$ ranging in D, and a set Γ of finitary relations on D we can associate a problem CSP(Γ), whose instances consist of a finite set of *constraints* of the form $(x_{i_1}, \ldots, x_{i_k}) \in R_j$ for some $R_j \in \Gamma$. The size of the instance (usually denoted by n) is by definition the number of different variables involved in its constrains.

As one might expect, for the tractability of CSP(Γ) the relations in Γ matter. For instance, general Boolean CSPs are *NP*-hard, but if all constraints are Horn clauses (i.e. disjunctions of literals, at most one of which is negative), then the problem is polynomially solvable. Other polynomially solvable cases include linear equations over finite fields and the set of all Boolean constraints that involve at most two variables.

The central question of the field is how the complexity of CSP(Γ) depends on Γ. Due to a beautiful result of Schaefer [46] we know, that in the Boolean case CSP(Γ) is either *NP*-hard or polynomial time solvable for every Γ. His *Dichotomy Theorem* also gives a full description of the polynomial time solvable families.

A fundamental question, raised by Feder and Vardi [21], asks if this theorem generalizes for arbitrary finite domain. Their *Dichotomy Conjecture* would imply the dichotomy of Monotone Monadic SNP ([21,34], see also [35]). This is perhaps the largest natural subclass of NP expected to have dichotomy. That the entire class NP does not have dichotomy (unless P=NP) was proved by Ladner [36].

In [21] it is established that it is sufficient to settle the dichotomy conjecture when Γ contains a single binary relation, i.e. a directed graph, H. With a slight abuse of notation we denote this problem by CSP(H). A problem instance now simply becomes a directed graph G whose vertices we want to map to the vertices of H such that edges go into edges. This is a graph homomorphism problem. What if G is undirected? In this case dichotomy holds by a pioneering theorem due to Hell and Nešetřil (1990):

Theorem 1 (*Hell–Nešetřil*)**.** *Assume that H is a simple, connected, undirected graph. Then CSP(H) is polynomial time solvable if and only if H is bipartite. Otherwise CSP(H) is NP-complete.*

Remark 2. The graph homomorphic view can be extended to arbitrary *relational structures*. Relational structures have a *type*, i.e. a list of relational names with associated arities. A relational structure of