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Fast recoloring of sparse graphs

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A B S T R A C T

In this paper, we show that for every graph of maximum average degree bounded away from d and any two $(d+1)$-colorings of it, one can transform one coloring into the other one within a polynomial number of vertex recolorings so that, at each step, the current coloring is proper. In particular, it implies that we can transform any 8-coloring of a planar graph into any other 8-coloring with a polynomial number of recolorings. These results give some evidence on a conjecture of Cereceda et al. (2009) which asserts that any $(d+2)$ coloring of a d-degenerate graph can be transformed into any other one using a polynomial number of recolorings.

We also show that any $(2d+2)$-coloring of a d-degenerate graph can be transformed into any other one with a linear number of recolorings.

1. Introduction

Reconfiguration problems consist in finding step-by-step transformations between two feasible solutions such that all intermediate states are also feasible. Such problems model dynamic situations where a given solution is in place and has to be modified, but no property disruption can be afforded. Recently, reconfiguration problems have raised a lot of interest in the context of constraint satisfaction problems [6,12] and of graph invariants like independent sets [13], dominating sets [3,15] or vertex colorings [4,5].

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In this paper $G = (V, E)$ is a graph where n denotes the size of V and k is an integer. For standard definitions and notations on graphs, we refer the reader to [10]. A "*proper*" *k-coloring* of G is a function $f : V(G) \to \{1, \dots, k\}$ such that, for every $xy \in E, f(x) \neq f(y)$. Throughout the paper we will only consider proper colorings. In the following, we will omit the proper for brevity. The *chromatic number* $\chi(G)$ of a graph G is the smallest k such that G admits a k-coloring. Two k-colorings are *adjacent* if they differ on exactly one vertex. The *k-recoloring graph of* G, denoted by $\mathcal{C}_k(G)$ and defined for any $k \geq \chi(G)$, is the graph whose vertices are k-colorings of G, with the adjacency condition defined above. Note that two colorings equivalent up to color permutation are distinct vertices in the recoloring graph. The graph G is *k-mixing* if $\mathcal{C}_k(G)$ is connected. Cereceda, van den Heuvel and Johnson provided an algorithm to decide whether, given two 3-colorings of a graph, one can transform the one into the other in polynomial time [8,9]. In particular, their result characterizes 3-mixing graphs. The easiest way to prove that a graph G is not k-mixing is to exhibit a *frozen* k-coloring of G, *i.e.* a coloring in which every vertex is adjacent to vertices of all other colors. Such a coloring is an isolated vertex in $\mathcal{C}_k(G)$.

Given any two colorings of a graph, to decide whether one can be transformed into the other, is **PSPACE**-complete for $k \geq 4$ [5]. The *k-recoloring diameter* of a k-mixing graph is the diameter of $\mathcal{C}_k(G)$. In other words, it is the minimum D for which any k-coloring can be transformed into any other one through a sequence of at most D adjacent k-colorings. Bonsma and Cereceda [5] proved that there exists a family of graphs and an integer k such that, for every graph G in the family there exist two k-colorings whose distance in the k-recoloring graph is finite and super-polynomial in n. However, the diameter of the k-recoloring may be polynomial when we restrict to a well-structured class of graphs and k is large enough. Graphs with bounded degeneracy are natural candidates.

The diameter of the k-recoloring graphs has been already studied in terms of the degeneracy of a graph. It was shown independently by Dyer et al. [11] and by Cereceda et al. [8] that for any $(d-1)$-degenerate graph G and every $k \geq d+1$, $\mathcal{C}_k(G)$ is connected $(\mathrm{diam}(\mathcal{C}_k(G)) < \infty)$. Moreover, Cereceda [7] also showed that for any $(d-1)$-degenerate graph G and every $k \geq 2d-1$, we have $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$.

Cereceda et al. conjectured in [8] that, for any $(d-1)$-degenerate graph G and every $k \geq d+1$, we have $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$. No general result is known so far on this conjecture, but several particular cases have been treated in the last few years. Bonamy et al. [4] showed that for every $(d-1)$-degenerate chordal graph and every $k \geq d+1$, $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$, improving the results of [8,11]. This result was then extended to graphs of bounded treewidth by Bonamy and Bousquet in [1]. Unfortunately, all these results are based on the existence of an underlying tree structure. This leads to nice proofs but new ideas are required to extend these results to other classes of graphs.

Our results. In Section 2, we show that Cereceda's quadratic bound on the recoloring diameter can be improved into a linear bound if one more color is available. More precisely we show that for every $(d-1)$-degenerate graph G and every $k \geq 2d$, the recoloring diameter of G is at most dn.

In Section 3, we study the k-recoloring diameter from another invariant of graphs related to degeneracy: the maximum average degree. The *maximum average degree* of G, denoted by $\mathrm{mad}(G)$, is the maximum average degree of a (non-empty) induced subgraph H of G. We prove that for every integer $d \geq 1$ and for every $\varepsilon > 0$, there exists $c = c(d, \varepsilon) \geq 1$ such that for every graph G satisfying $\mathrm{mad}(G) \leq d - \varepsilon$ and for every $k \geq d+1$, $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^c)$. The proof goes as follows. We first show that the vertex set can be partitioned into a logarithmic number of sparse sets. Using this partition, we show that one color can be eliminated after a polynomial number of recolorings and then we finally conclude by an iterative argument.

Since every planar graph G satisfies $\mathrm{mad}(G) \leq 6$, our result implies that for every $k \geq 8$ the diameter of the k-recoloring graph of G is polynomial in n. Bousquet and Bonamy observed in [2] that $k \geq 7$ is needed to obtain such a conclusion and conjectured that $k = 7$ is enough (this is the planar graph version of the conjecture raised by Cereceda et al. [8] for degenerated graphs). We also discuss the limitations of our approach by showing that it cannot provide a polynomial bound on the diameter of the 7-recoloring graph of a planar graph. Finally, we also mention other consequences of our result to triangle-free planar graphs.

The degeneracy is closely related to the maximum average degree: a graph G satisfying $\mathrm{mad}(G) \leq d$ is d-degenerate and every d-degenerate graph has maximum average degree at most $2d$ (see e.g.