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How to hunt an invisible rabbit on a graph

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ABSTRACT

We investigate Hunters & Rabbit game on graphs, where a set of hunters tries to catch an invisible rabbit that is forced to slide along an edge of a graph at every round. We show that the minimum number of hunters required to win on an $(n\times m)$-grid is $\lfloor \frac{\min\{n,m\}}{2} \rfloor + 1$. We also show that the extremal value of this number on n-vertex trees is between $\Omega(\log n/\log\log n)$ and $O(\log n)$.

1. Introduction

Our work originated from the following game puzzle. Hunter wants to shoot Rabbit who is hiding behind one of the three bushes growing in a row. Hunter does not see Rabbit, so he select one of the bushes and shoots at it. If Rabbit is behind the selected bush, then Hunter wins. Otherwise Rabbit, scared by the shot, jumps to one of the adjacent bushes. As Rabbit is infinitely fast, Hunter sees neither Rabbit's old nor new bush and has to select where to shoot again. Can Hunter always win in this game?

Of course, the answer is *yes*: Hunter has to shoot twice at the middle bush. If he misses the first time, it means that Rabbit was hiding either behind the leftmost or the rightmost bush. In both cases, the only adjacent bush where Rabbit can jump after the first shot is the middle one, thus the second shot at the middle bush finishes the game. A natural question is what happens if we have four bushes, and more generally, $n \geq 3$ bushes growing in a row? After a bit of thinking, the answer here is *yes* as

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well. This time Hunter wins by shooting consequently at the bushes $2, \ldots, n-1$ when n is odd and at the bushes $2, \ldots, n$ when n is even, and then repeating the same sequence of shots again.

In a slightly different situation, when bushes grow around a circle, say we have three bushes and Rabbit can jump from any of them to any of them, then Hunter cannot guarantee the success anymore. In this situation we need the second hunter and this brings us to the following setting. We consider Hunters & Rabbit game with two players, Hunter and Rabbit, playing on an undirected graph. Hunter player has a team of hunters who attempt to shoot the rabbit. At the beginning of the game, Rabbit player selects a vertex and occupies it. Then the players take turns starting with Hunter player. At every round of the game each of the hunters selects some vertex of the graph and the hunters shoot simultaneously at their respective aims. If the rabbit is not in a vertex that is hit by a shot, it jumps to an adjacent vertex. The rabbit is invisible to the hunters, but since we are interested in the guaranteed success of the hunters, we can assume that rabbit has a complete knowledge about all shots that the hunters plan. Hunter player wins if at some round of the game he succeeds to shoot the rabbit, and Rabbit player wins if the rabbit can avoid these situations forever. For a given graph G, we are interested in the minimum number of hunters sufficient to win in the Hunters & Rabbit game on G, for any strategy chosen by the rabbit player. We call this parameter the *hunting number* of a graph, and denote it by $h(G)$.

Related work. Britnell and Wildon studied the case with one hunter in [4]. They characterized the graphs for which one hunter (the prince in their terminology) can find the rabbit (the princess). This result was also independently discovered by Haslegrave [9], who not only characterized the graphs with hunting number one (in cat and mouse terminology) but also provided best possible capture times on such graphs. This problem is also mentioned as problem 6* in [6, p. 4] (as a problem of shooting shelters connected by tunnels) with a full solution given on pp. 52–54.

Hunters & Rabbit game is closely related to several pursuit-evasion and search games on graphs, see [7] for further references. In pursuit-evasion games a team of cops is trying to catch a robber located on the vertices of the graph. In cops–robbers terminology, Hunters & Rabbit is the Cops & Robber game, where the set of cops on helicopters (that is allowed to jump to any vertex) is trying to catch an invisible robber. The robber moves only to adjacent vertices and is forced to move every time the cops are in the air.

In particular, the classical Cops & Robbers games introduced independently by Winkler and Nowakowski [12] and by Quilliot [13] (see also the book by Bonato and Nowakowski [2] for the detailed introduction to the field), is the game where robber is visible, and the cops and robber move to adjacent vertices or remain on their present vertex. The variant of the game where the robber is invisible introduced by Tošić [14] and the variant where the cops use predefined paths as theirs search moves was introduced by Brass et al. [3]. Another related search game, node search, was introduced by Kirousis and Papadimitriou in [10,11]. Here cops can fly, that is move to any vertex they wish, the robber is invisible and very fast, that is can go to any vertex connected to his current location by a path containing no cops. Thus, Hunters & Rabbit can be seen as a variant of Tošić's game where cops have more power or as a variant of Kirousis–Papadimitriou's game, where the robber is more restricted. One more significant difference with mentioned games is that in most versions of Cops & Robbers games the robber is not forced to move at every round of the game, while in our setting the rabbit cannot stay at the same vertex for two consecutive rounds.

A randomized game called Hunter vs. Rabbit was considered by Adler et al. [1]; here, the hunter is allowed to move only along edges of the graph while there are no constraints on rabbit's moves.

Our results and organization of the paper. The remaining part of this paper is organized as follows. We give basic definitions and preliminary results in Section 2. We also show in this section that the hunting number of a graph does not exceed its pathwidth plus one and the bound is tight. In Section 3, we prove our first main result, namely that for an $(n \times m)$-grid G, it holds that $h(G) = \lfloor \frac{\min\{n,m\}}{2} \rfloor + 1$. This result is based on a new isoperimetric theorem that we find interesting on its own. In Section 4, we provide bounds on the hunting number of trees, which is the second contribution of the paper. We show that the hunting number of an n-vertex tree is always $O(\log n)$, but there are trees where it can be as large as $\Omega(\log n / \log \log n)$. We conclude with open problems in Section 5.