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Decomposing plane cubic graphs

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ABSTRACT

It was conjectured by Hoffmann-Ostenhof that the edge set of every cubic graph can be decomposed into a spanning tree, a matching and a family of cycles. We prove the conjecture for 3-connected cubic plane graphs and 3-connected cubic graphs on the projective plane. Our proof provides a polynomial time algorithm to find the decomposition for 3-connected cubic plane graphs.

1. Introduction

All graphs discussed in this paper are simple. A graph G consists of the vertex set $V(G)$ and the edge set $E(G)$. A graph G is *cubic* if every vertex v in G has degree 3. A graph without cycle is called an *acyclic* graph or a forest. A *spanning tree* of a graph G is a connected acyclic subgraph containing all vertices of G. A *matching* is a set of edges without common end vertices. A matching is *perfect* if it covers all vertices of G.

A *decomposition* of a graph G consists of pairwise edge-disjoint subgraphs whose union is G, that is, each edge in G belongs to exactly one of the subgraphs. The decompositions of graphs to forests and degree-bounded subgraphs have applications in graph coloring (cf. [5,15]). In [14], Gonçalves proved that every plane graph has a decomposition into three forests one of which has degree at most 4, which was conjectured by Balogh, Kochol, Pluhar and Yu in [3]. Kleitman [19] proved that a plane graph with girth at least 6 has a decomposition into a forest, pairwise edge-disjoint paths and cycles. Further, a plane graph with large girth (at least 8) has a decomposition into a forest and a matching [5,15,28].

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But a plane graph with smaller girth does not have these decompositions [19,24]. The decomposition problem for sparse graphs also has been studied in [18,24].

For decompositions of cubic graphs with certain properties, the first result is the Vizing Theorem [27] on proper edge-coloring, which indicates that every cubic graph has a decomposition into four pairwise edge-disjoint matchings. Recently, Fouquet and Vanherpe studied the decomposition of cubic graphs into pairwise edge-disjoint paths with certain properties [12,11]. As pointed out in [12], the decomposition problem of cubic graphs into paths is related to conjectures on cubic graphs, for example, the Fan–Raspaud conjecture [9] (which states that every 2-edge-connected cubic graph contains three perfect matchings with empty intersection). Note that every connected graph G with an even number of edges can be decomposed into pairwise edge-disjoint paths of length exactly 2. (To see this, consider the line graph $L(G)$ of G, which is a connected claw-free graph with an even number of vertices and hence has a perfect matching (see [21,25]). A perfect matching of $L(G)$ corresponds to a desired decomposition of G).

A cubic graph does not have a decomposition into a forest and a matching because of the degree condition. But the Petersen Theorem implies that every 2-connected cubic graph can be decomposed into a forest (a perfect matching) and a family of cycles (a 2-factor). It seems also interesting to consider a decomposition of a cubic graph into a spanning tree and other subgraphs. A spanning tree T is called a *homeomorphically irreducible spanning tree* or shortly a HIST if T does not contain a vertex of degree 2 (see [2]). A cubic graph with a HIST is equivalent to having a decomposition into a spanning tree and a family of cycles. Malkevitch [22] investigated HIST in 3-polytopes and found infinitely many 3-connected cubic plane graphs without a HIST (see also examples on Page 81 in [8] or consider the prism over cycles). Albertson, Berman, Hutchinson and Thomassen [2] asked the following question: for each k, is there a cyclically k-edge-connected cubic graph without a HIST? Recently this question was shown by Hoffmann-Oftenhof and Ozeki [17] to be positive, (that is, for each $k \geq 4$, there is a cyclically k-edge-connected cubic graph without a HIST). Douglas [7] show that it is NP-complete to determine whether a given plane graph with maximum degree 3 has a HIST or not. Instead of HIST, Hoffmann-Ostenhof made the following conjecture for all connected cubic graphs.

Conjecture 1.1 (*Hoffmann-Ostenhof*)**.** *Let G be a connected cubic graph. Then G has a decomposition into a spanning tree, a matching and a family of cycles.*

Conjecture 1.1 first appeared in [16] (see also [6, Problem BCC 22.12] and [20]). There are a few partial results known for Conjecture 1.1. Kostochka [20] noticed that the Petersen graph, the prisms over cycles, and many other graphs have a decomposition desired in Conjecture 1.1. Akbari [1] showed that Conjecture 1.1 is true for Hamiltonian cubic graphs.

In this paper, we prove Conjecture 1.1 for 3-connected cubic plane graphs. The following is our main theorem.

Theorem 1.2. *Let G be a 3-connected cubic plane graph. Then G can be decomposed into a spanning tree, a matching and a family of cycles.*

Note that a 3-connected cubic plane graph does not necessarily have a Hamiltonian cycle (see [26]) and a HIST (see the above). In the next section, we show a slightly stronger result (Theorem 2.1) than Theorem 1.2. The proof of Theorem 2.1 provides a polynomial time algorithm to find the decomposition. As another consequence of Theorem 2.1, we have the following result for cubic graphs on the projective plane. A proof of Theorem 1.3 is given in Section 3.

Theorem 1.3. *Let G be a 3-connected cubic graph embedded in the projective plane. Then G has a decomposition into a spanning tree, a matching and a family of cycles.*

2. Proof of Theorem 1.2

Let G be a connected plane graph. We denote the outer facial walk of G by ∂G. A facial cycle F of G is said to be *second outer* if F and ∂G shares at least one edge and $F \neq \partial G$. For two vertices u and v in