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Maximum density of induced 5-cycle is achieved by an iterated blow-up of 5-cycle

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ABSTRACT

Let $C(n)$ denote the maximum number of induced copies of 5-cycles in graphs on n vertices. For n large enough, we show that $C(n) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e)$, where $a + b + c + d + e = n$ and a, b, c, d, e are as equal as possible.

Moreover, for n a power of 5, we show that the unique graph on n vertices maximizing the number of induced 5-cycles is an iterated blow-up of a 5-cycle.

The proof uses flag algebra computations and stability methods.

1. Introduction

In 1975, Pippenger and Golumbic [20] conjectured that in graphs the maximum induced density of a k-cycle is $k!/(k^k - k)$ when $k \geq 5$. In this paper we solve their conjecture for $k = 5$. In addition, we also show that the extremal limit object is unique. The problem of maximizing the induced density of C_5 is also posted on http://flagmatic.org as one of the problems where the plain flag algebra method was applied but failed to provide an exact result. It was also mentioned by Razborov [25].

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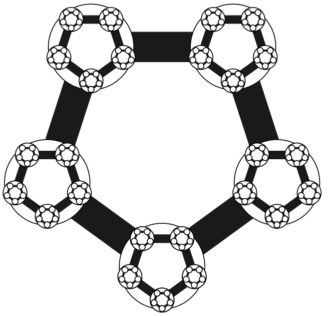

Fig. 1. The graph $C_5^{k\times}$ maximizes the number of induced C_5s.

Problems of maximizing the number of induced copies of a fixed small graph H have attracted a lot of attention recently [8,14,29]. For a list of other results on this so called inducibility of small graphs of order up to 5, see the work of Even-Zohar and Linial [8].

Denote the $(k-1)$-times iterated blow-up of C_5 by $C_5^{k\times}$, see Fig. 1. Let $\mathcal{G}_n$ be the set of all graphs on n vertices, and denote by $C(G)$ the number of induced copies of C_5 in a graph G. Define

$$C(n) = \max_{G \in \mathcal{G}_n} C(G).$$

We say a graph $G \in \mathcal{G}_n$ is *extremal* if $C(G) = C(n)$. Notice that, since C_5 is a self-complementary graph, G is extremal if and only if its complement is extremal. If n is a power of 5, we can exactly determine the unique extremal graph and thus $C(n)$.

Theorem 1. *For $k \geq 1$, the unique extremal graph in $\mathcal{G}_{5^k}$ is $C_5^{k\times}$.*

To prove Theorem 1, we first prove the following theorem. Note that this theorem is sufficient to determine the unique limit object (the graphon) maximizing the density of induced copies of C_5.

Theorem 2. *There exists n_0 such that for every $n \geq n_0$*

$$C(n) = a \cdot b \cdot c \cdot d \cdot e + C(a) + C(b) + C(c) + C(d) + C(e),$$

where $a + b + c + d + e = n$ and a, b, c, d, e are as equal as possible.

Moreover, if $G \in \mathcal{G}_n$ is an extremal graph, then $V(G)$ can be partitioned into five sets X_1, X_2, X_3, X_4, and X_5 of sizes a, b, c, d and e respectively, such that for $1 \leq i < j \leq 5$ and $x_i \in X_i$, $x_j \in X_j$, we have $x_i x_j \in E(G)$ if and only if $j - i \in \{1, 4\}$.

In Section 2, we give a brief overview of our method, in Section 3 we prove Theorem 2, and in Section 4 we prove Theorem 1.

2. Method and flag algebras

Our method relies on the theory of flag algebras developed by Razborov [21]. Flag algebras can be used as a general tool to attack problems from extremal combinatorics. Flag algebras were used for a wide range of problems, for example the Caccetta–Häggkvist conjecture [15,24], Turán-type problems in graphs [7,11,13,19,22,26,27], 3-graphs [9,10] and hypercubes [1,3], extremal problems in a colored environment [2,4,6], and also to problems in geometry [17] or extremal theory of permutations [5]. For more details on these applications, see a recent survey of Razborov [23].

A typical application of the so-called *plain flag algebra method* provides a bound on densities of substructures. To get a good bound, true inequalities and equalities involving the densities of substructures are combined with the help of semidefinite programming. This step is by now largely automated, there is even an open source application called Flagmatic [29], which gives easy to check certificates for the validity of this step. In some cases the bound is asymptotically sharp. Obtaining