European Journal of Combinatorics 52 (2016) 74–78

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Cycles with consecutive odd lengths

Jie Ma

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, China

ARTICLE INFO

Article history:
Received 1 October 2014
Accepted 9 September 2015
Available online 8 October 2015

ABSTRACT

In this paper we prove that there exists an absolute constant $c > 0$ such that for every natural number k, every non-bipartite 2-connected graph with average degree at least ck contains k cycles with consecutive odd lengths. This implies the existence of the absolute constant $d > 0$ that every non-bipartite 2-connected graph with minimum degree at least dk contains cycles of all lengths modulo k, thus providing an answer (in a strong form) to a question of Thomassen in Thomassen (1983). Both results are sharp up to the constant factors.

1. Introduction

The research of cycles has been fundamental since the beginning of graph theory. One of various problems on cycles which have been considered is the study of cycle lengths modulo a positive integer k. Burr and Erdős [6] conjectured that for every odd k, there exists a constant c_k such that every graph with average degree at least c_k contains cycles of all lengths modulo k. In [2], Bollobás resolved this conjecture by showing that $c_k = 2[(k+1)^k - 1]/k$ suffices. Thomassen [14,15] strengthened the result of Bollobás by proving that for every k (not necessarily odd), every graph with minimum degree at least $4k(k+1)$ contains cycles of all even lengths modulo k, which was improved to the bound $2k-1$ by Diwan [5]. Note that in case k is even, any integer congruent to l modulo k has the same parity with l, and thus we cannot expect that there are cycles of all lengths modulo k in bipartite graphs (even with sufficient large minimum degree). On the other hand, Thomassen [14] showed that for every k there exists a least natural number $f(k)$ such that every non-bipartite 2-connected graph with minimum degree at least $f(k)$ contains cycles of all lengths modulo k. Here the 2-connectivity condition cannot be further improved, as one can easily construct a non-bipartite connected graph with arbitrarily large

E-mail address: jiema@ustc.edu.cn.

http://dx.doi.org/10.1016/j.ejc.2015.09.002
0195-6698/

minimum degree but containing a unique (also arbitrary) odd cycle. Thomassen [14] remarked that the upper bound for $f(k)$ obtained by him is perhaps "far too large" and asked if $f(k)$ can be bounded above by a polynomial.

Bondy and Vince [3] resolved a conjecture of Erdős by showing that every graph with minimum degree at least 3 contains two cycles whose lengths differ by one or two. Verstraëte [17] proved that if a graph G has average degree at least $8k$ and even girth g then there are $(g/2-1)k$ cycles of consecutive even lengths in G. In an attempt to extend the result of Bondy and Vince, Fan [9] showed that every graph with minimum degree at least $3k-2$ contains k cycles of consecutive even lengths or consecutive odd lengths. Sudakov and Verstraëte proved [11] that if a graph G has average degree $192(k+1)$ and girth g then there are $k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive even lengths in G, strengthening the above results in the case that k and g are large. It is natural to ask if one can pursue the analogous result for odd cycles. In this paper, we show that this indeed is the case by the following theorem.

Theorem 1. *There exists an absolute constant $c > 0$ such that for every natural number k, every non-bipartite 2-connected graph G with average degree at least ck and girth g contains at least $k^{\lfloor (g-1)/2 \rfloor}$ cycles of consecutive odd lengths.*

We shall show that $c = 456$ suffices. We point out that the non-bipartite condition here is necessary and the 2-connectivity condition cannot be improved. In view of the *Moore Bound*, our lower bound on the number of cycles is best possible (up to constant factor) for infinitely many integers k when $g \le 8$ or $g = 12$. And more generally, the well-known conjecture that the minimal order of graphs with minimal degree k and girth g is $O(k^{\lfloor (g-1)/2 \rfloor})$ would imply that our result gives the correct order of magnitude for other values of g. More details about this conjecture can be found in the survey by Exoo and Jajcay [8].

Let G be a graph as in Theorem 1. It is clear that there are at least k cycles of consecutive odd lengths in G, which assures that G contains cycles of all odd lengths modulo k (whenever k is even or odd). Together with the aforementioned result of Diwan on cycles of all even lengths modulo k, we answer the question of Thomassen by improving the upper bound of $f(k)$ to a linear function by the following corollary.

Corollary 2. *There exists an absolute constant $d > 0$ such that for every natural number k, every non-bipartite 2-connected graph with minimum degree at least dk contains cycles of all lengths modulo k.*

This bound is sharp up to the constant factor: the complete graph K_{k+1} contains cycles of all lengths but 2 modulo k and thus shows that $f(k) \ge k+1$.

All graphs considered are simple and finite. Let G be a graph. We denote the number of vertices in G by $|G|$, the vertex set by $V(G)$, the edge set by $E(G)$ and the minimum degree by $\delta(G)$. If $S \subset V(G)$, then $G-S$ denotes the subgraph of G obtained by deleting all vertices in S (and all edges incident with some vertex in S). If $S \subset E(G)$, then $G-S$ is obtained from G by deleting all edges in S. Let A and B be subsets of $V(G)$. An *(A, B)-path* in G is a path with one endpoint in A and the other in B. If A only contains one vertex a, then we simply write (A, B)-path as (a, B)-path. We say a path P is *internally disjoint* from A, if no vertex except the endpoints in P is contained in A.

The rest of this paper is organized as follows. In Section 2, we establish Theorem 1 based on the approach of [11]. Section 3 contains some remarks and open problems. We make no effort to optimize the constants in the proofs and instead aim for simpler presentation.

2. The proof

Before presenting the proof, we state the following useful lemma from [17], which will be applied multiple times and become essential in the proof of our main theorem. By *a cycle with a chord*, we mean the union of a cycle and a chord of this cycle.

Lemma 3 (*Verstraëte [17, Lemma 2]*)**.** *Let C be a cycle with a chord, and let (A, B) be a nontrivial partition of $V(C)$. Then C contains (A, B)-paths of every length less than $|C|$, unless C is bipartite with bipartition (A, B).*