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# Intersection theorems for multisets

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ABSTRACT

Let $k$, $t$ and $m$ be positive integers. A $k$-multiset of $[m]$ is a collection of $k$ integers from the set $\{1, \ldots, m\}$ in which the integers can appear more than once. We use graph homomorphisms and existing theorems for intersecting and $t$-intersecting $k$-set systems to prove new results for intersecting and $t$-intersecting families of $k$-multisets. These results include a multiset version of the Hilton–Milner theorem and a theorem giving the size and structure of the largest $t$-intersecting family of $k$-multisets of an $m$-set when $m \geq 2k - t$.



## 1. Introduction

In this paper, we show that the method used in [14] to prove a natural extension of the famous Erdős–Ko–Rado theorem to multisets can be used to prove additional intersection theorems for multisets.

We prove a multiset version of the Hilton–Milner theorem; this result gives the largest family of intersecting multisets satisfying the condition that the intersection of all of the multisets in the family is empty. We determine the largest family of $k$-multisets having the property that no more than $s$ of the multisets from the family can be pairwise disjoint. A related question that we answer is, what is the largest family of multisets that can be partitioned into two intersecting families? We also consider multisets that have intersection of size at least $t$, so are $t$-intersecting. We prove a theorem giving the size and structure of the largest $t$-intersecting family of $k$-multisets of an $m$-set when $m \geq 2k - t$. Finally, we prove a version of the Hilton–Milner theorem for $t$-intersecting multisets.

In Section 1, we introduce notation and provide some background information on the Erdős–Ko–Rado theorem and the known results for intersecting families of multisets. Additional known

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results for intersecting set systems are stated in Section 2. In Section 3, we extend the results from Section 2 to families of multisets. Section 4 contains our results concerning $t$-intersecting families of multisets. In Section 5, we discuss some open problem for multisets.

### 1.1. Notation and definitions

Throughout this paper, small letters are used to denote integers, capital letters are used for sets (and multisets) of integers, and script capital letters are used for collections of sets or multisets. The set of integers from $x$ to $y$ inclusive is represented by $[x, y]$. If $x = 1$, this is simplified to $[y]$.

A $k$-set (or $k$-subset) is a set of cardinality $k$ and the collection of all $k$-subsets of $[n]$ is denoted by $\binom{[n]}{k}$. We say that a collection of sets is *intersecting* if every pair of sets in the collection is intersecting and that it is $t$-*intersecting* if every pair has at least $t$ elements in common. Collections are said to be *isomorphic* if one can be obtained from the other by a permutation of the underlying set.

### 1.2. Background

The Erdős–Ko–Rado theorem is an important result in extremal set theory that gives the size and structure of the largest intersecting $k$-subset system from $[n]$. It appeared in a paper published in 1961 [5] which contains two main theorems. The first of these is commonly stated as follows:

**Theorem 1.1** (*Erdős, Ko and Rado [5]*). *Let $k$ and $n$ be positive integers with $n \geq 2k$. If $\mathcal{F}$ is a collection of intersecting $k$-subsets of $[n]$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

*Moreover, if $n > 2k$, equality holds if and only if $\mathcal{F}$ is a collection of all the $k$-subsets from $[n]$ that contain a fixed element from $[n]$.*

A Kneser graph, denoted by $K(n, k)$, is a graph whose vertices are the $k$-subsets of $[n]$. Two vertices are adjacent if and only if the corresponding $k$-subsets are disjoint. Thus an independent set of vertices is an intersecting set system and the cardinality of the largest independent set is equal to $\binom{n-1}{k-1}$. (It is assumed that $n \geq 2k$ since otherwise the graph would be the empty graph.)

A second theorem in [5] gives the size and structure of the largest $t$-intersecting $k$-subset system provided that $n$ is sufficiently large relative to $k$ and $t$. A later theorem due to Ahlswede and Khachatrian extends this result to all values of $n$, $k$ and $t$. In virtually all cases, the families attaining maximum size are isomorphic to $\mathcal{F}_{(r)}$, where $\mathcal{F}_{(r)}$ is defined as follows. For $t, k, n \in \mathbb{N}$, let

$$\mathcal{F}_{(r)} = \left\{ A \in \binom{[n]}{k} : |A \cap [t+2r]| \geq t+r \right\}.$$

In the following statement of the theorem, it is assumed that $n > 2k - t$ since for $n \leq 2k - t$, the collection of all $k$-subsets of $[n]$ is $t$-intersecting.

**Theorem 1.2** (*Ahlswede and Khachatrian [1]*). *Let $t$, $k$ and $n$ be positive integers with $t \leq k \leq n$ and let $r$ be a non-negative integer such that $r \leq k - t$.*

1. *If*

$$(k-t+1)\left(2+\frac{t-1}{r+1}\right) < n < (k-t+1)\left(2+\frac{t-1}{r}\right),$$

   *then $\mathcal{F}_{(r)}$ is the unique (up to isomorphism) $t$-intersecting $k$-set system with maximum size. (By convention, $\frac{t-1}{r} = \infty$ for $r = 0$.)*
2. *If $n = (k-t+1)(2+\frac{t-1}{r+1})$, then $\left|\mathcal{F}_{(r)}\right| = \left|\mathcal{F}_{(r+1)}\right|$ and a system of maximum size will equal (up to isomorphism) $\mathcal{F}_{(r)}$ or $\mathcal{F}_{(r+1)}$.*