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An improved incidence bound for fields of prime order

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ABSTRACT

Let P be a set of points and L a set of lines in $\mathbb{F}_p^2$, with $|P|, |L| \leq N$ and $N < p$. We show that P and L generate no more than $CN^{\frac{3}{2}-\frac{1}{806}+o(1)}$ incidences for some absolute constant C. This improves on the previously best-known bound of $CN^{\frac{3}{2}-\frac{1}{10,678}}$.

1. Introduction

Throughout this paper we use $X = \Omega(Y)$, $Y = O(X)$, and $Y \ll X$ all to mean that there is an absolute constant C with $Y \leq CX$. We shall write $X \approx Y$ if $X \ll Y$ and $Y \ll X$.

1.1. Incidences

This paper is about counting incidences between points and lines in a plane. A point is *incident* to a line if it lies on that line. Incidences are counted with multiplicity, in the sense that several lines incident to the same point determine several incidences, and vice versa.

We are interested in knowing the maximum number of incidences between a set P of points and a set L of lines, say with $|P|, |L| \leq N$. Certainly this cannot exceed N^2. But using the Cauchy–Schwarz inequality and the fact that two distinct points determine a line, it is straightforward to see that it is in fact $O(N^{3/2})$.

So, writing $I(P, L)$ for the number of incidences between P and L, *non-trivial* incidence bounds are of the form $I(P, L) = O\left(N^{3/2-\epsilon}\right)$ with $\epsilon > 0$. The larger the value of ϵ, the stronger the bound.

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A natural example shows that the strongest result that can be hoped for is $\epsilon = 1/6$. Progress towards achieving this depends on the ambient field over which points and lines are defined. In the case of the plane $\mathbb{R}^2$, the best possible result was obtained by Szemerédi and Trotter [13]:

Theorem 1 (*Szemerédi–Trotter*)**.** *Let P be a set of points and L a set of lines in $\mathbb{R}^2$ with $|P|, |L| \leq N$. Then $I(P, L) = O(N^{4/3})$.*

This result was generalised to $\mathbb{C}^2$ by Tóth [15], and a near-sharp generalisation to higher dimensional points and varieties was recently given by Solymosi and Tao [12].

When working over finite fields, some additional condition must be imposed if we are to prove nontrivial incidence bounds. Otherwise we would be free to take P to be the entire plane, where only trivial bounds would be possible.

Vinh [16] has given a nontrivial incidence bound for sets that are a large part, but not all of, of the plane.

Theorem 2 (*Vinh*)**.** *Let $\mathbb{F}_q$ be the finite field of order q. Let P be a set of points and L a set of lines in $\mathbb{F}_q^2$ with $|P|, |L| \leq N$ and $p^{1+\gamma} < N < p^{2-\gamma}$ for $0 < \gamma < 1$. Then $I(P, L) = O(N^{\frac{3}{2}-\frac{\gamma}{4}})$.*

In the case of smaller sets, Helfgott and Rudnev [4] obtained the following result for the finite field of prime order p.

Theorem 3 (*Helfgott–Rudnev*)**.** *Let $\mathbb{F}_p$ be the finite field of prime order p. Let P be a set of points and L a set of lines in $\mathbb{F}_p^2$ with $|P|, |L| \leq N$ and $N < p$. Then $I(P, L) = O(N^{\frac{3}{2}-\frac{1}{10,678}})$.*

This followed work of Bourgain, Katz and Tao [2], which established the existence of a non-zero ϵ so long as $N < p^{2-\delta(\epsilon)}$, but did not quantify it. The present author [5] extended Theorem 3 to a finite field $\mathbb{F}_q$ of general order, subject to analogous conditions to prevent P from being a large part of a subplane, and with a slightly weaker exponent.

This paper proves the following theorem, which improves on Theorem 3:

Theorem 4. *Let $\mathbb{F}_p$ be the finite field of prime order p. Let P be a set of points and L a set of lines in $\mathbb{F}_p^2$ with $|P|, |L| \leq N$ and $N < p$. Then $I(P, L) = O(N^{\frac{3}{2}-\frac{1}{806}+o(1)})$.*

As will be seen, the proof of Theorem 4 uses finite field sum–product estimates as a 'black box'. It is likely that further improvements can be obtained by unpacking the sum–product proof and exploiting its use of multiplicative energy and covering arguments to bypass some costly Balog–Szemerédi–Gowers type refinements. It should also be remarked that by combining recent sum–product work of Li and Roche-Newton [7] with the approach in [5], a result comparable to Theorem 4 should hold for a general finite field $\mathbb{F}_q$, subject to appropriate non-degeneracy conditions.

1.2. Relation to sum–product estimates

If A and B are subsets of a field F, then we will write

$$A + B = \{a + b : a \in A, b \in B\}.$$
$$A \cdot B = \{ab : a \in A, b \in B\}.$$

These definitions extend analogously to subtraction and division.

Erdős and Szemerédi conjectured that any finite set $A \subseteq \mathbb{R}$ must satisfy

$$\max\{|A + A|, |A \cdot A|\} \gg_\epsilon |A|^{2-\epsilon}$$

for any $\epsilon > 0$. Nontrivial lower bounds on $\max\{|A + A|, |A \cdot A|\}$ are generally called *sum–product estimates*.

The strongest-known result is due to Solymosi [11], who obtained

$$\max\{|A + A|, |A \cdot A|\} \gg |A|^{\frac{4}{3}-o(1)}.$$