European Journal of Combinatorics 52 (2016) 146–173

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Complexity of short rectangles and periodicity

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ARTICLE INFO

Article history:
Received 15 October 2014
Accepted 9 October 2015
Available online 1 November 2015

ABSTRACT

The Morse–Hedlund Theorem states that a bi-infinite sequence η in a finite alphabet is periodic if and only if there exists $n \in \mathbb{N}$ such that the block complexity function $P_\eta(n)$ satisfies $P_\eta(n) \leq n$. In dimension two, Nivat conjectured that if there exist $n, k \in \mathbb{N}$ such that the $n \times k$ rectangular complexity $P_\eta(n, k)$ satisfies $P_\eta(n, k) \leq nk$, then η is periodic. Sander and Tijdeman showed that this holds for $k \leq 2$. We generalize their result, showing that Nivat's Conjecture holds for $k \leq 3$. The method involves translating the combinatorial problem to a question about the nonexpansive subspaces of a certain $\mathbb{Z}^2$ dynamical system, and then analyzing the resulting system.

1. Nivat's conjecture for colorings of height 3

1.1. Background and statement of the theorem

The Morse–Hedlund Theorem [8] gives a classic relation between the periodicity of a bi-infinite sequence taking values in a finite alphabet $\mathcal{A}$ and the complexity of the sequence. For higher dimensional sequences $\eta = \left(\eta(\vec{n}) : \vec{n} \in \mathbb{Z}^d\right)$ with $d \geq 1$ taking values in the finite alphabet $\mathcal{A}$, a possible generalization is the Nivat Conjecture [9]. To state this precisely, we define $\eta : \mathbb{Z}^d \to \mathcal{A}$ to be *periodic* if there exists $\vec{m} \in \mathbb{Z}^d$ with $\vec{m} \neq \vec{0}$ such that $\eta(\vec{n} + \vec{m}) = \eta(\vec{n})$ for all $\vec{n} \in \mathbb{Z}^d$ and define the *rectangular complexity* $P_\eta(n_1, \ldots, n_d)$ to be the number of distinct $n_1 \times \cdots \times n_d$ rectangular colorings that occur in η. Nivat conjectured that for $d = 2$, if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk$, then η is periodic. This is a two dimensional phenomenon, as counterexamples for the corresponding

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http://dx.doi.org/10.1016/j.ejc.2015.10.003
0195-6698/

statement in dimension $d \geq 3$ were given in [11]. There are numerous partial results, including for example [11,6,10] (see also related results in [1,3,5]). In [4] we showed that under the stronger hypothesis that there exist $n, k \in \mathbb{N}$ such that $P_\eta(n, k) \leq nk/2$, then η is periodic.

We prove that Nivat's Conjecture holds for rectangular colorings of height at most 3:

Theorem 1.1. *Suppose $\eta: \mathbb{Z}^2 \to \mathcal{A}$, where $\mathcal{A}$ denotes a finite alphabet. Assume that there exists $n \in \mathbb{N}$ such that $P_\eta(n, 3) \leq 3n$. Then η is periodic.*

If there exists $n \in \mathbb{N}$ such that $P_\eta(n, 1) \leq n$, periodicity of η follows quickly from the Morse–Hedlund Theorem [8]: each row is horizontally periodic of period at most n and so $n!$ is an upper bound for the minimal horizontal period of η. When there exists $n \in \mathbb{N}$ such that $P_\eta(n, 2) \leq 2n$, periodicity of η was established by Sander and Tijdeman [12]. The extension to colorings of height 3 is the main result of this article. By the obvious symmetry, the analogous result holds if there exists $n \in \mathbb{N}$ such that $P_\eta(3, n) \leq 3n$.

1.2. Generalized complexity functions

To study rectangular complexity, we need to consider the complexity of more general shapes. As introduced by Sander and Tijdeman [11], if $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set, we define $P_\eta(\mathcal{S})$ to be the number of distinct colorings in η that can fill the shape $\mathcal{S}$. For example, $P_\eta(n, k) = P_\eta(R_{n,k})$, where $R_{n,k} = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x < n, 0 \leq y < k\}$. Similar to methods introduced in [4], we find subsets of $R_{n,3}$ (the *generating sets*) that can be used to study periodicity. Using the restrictive geometry imposed by colorings of height 3, we derive stronger properties that allow us to prove periodicity only using the complexity bound $3n$, rather than $3n/2$ as relied upon in [4].

1.3. Translation to dynamics

As in [4], we translate the problem to a dynamical one. We define a dynamical system associated with $\eta: \mathbb{Z}^2 \to \mathcal{A}$ in a standard way: endow $\mathcal{A}$ with the discrete topology, $X = \mathcal{A}^{\mathbb{Z}^2}$ with the product topology, and define the $\mathbb{Z}^2$-action by translations on X by $(T^{\vec{u}}\eta)(\vec{x}) := \eta(\vec{x} + \vec{u})$ for $\vec{u} \in \mathbb{Z}^2$. With respect to this topology, the maps $T^{\vec{u}}: X \to X$ are continuous. Let $\mathcal{O}(\eta) := \{T^{\vec{u}}\eta : \vec{u} \in \mathbb{Z}^2\}$ denote the $\mathbb{Z}^2$-orbit of $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and set $X_\eta := \overline{\mathcal{O}(\eta)}$. When we refer to the dynamical system X_η, we implicitly assume that this means the space X_η endowed with the $\mathbb{Z}^2$-action by the translations $T^{\vec{u}}$, where $\vec{u} \in \mathbb{Z}^2$. Note that in general $\overline{\mathcal{O}(\eta)} \setminus \mathcal{O}(\eta)$ is nonempty.

The dynamical system X_η reflects the properties of η. An often used fact is that if $F \subset \mathbb{Z}^2$ is finite and $f \in X_\eta$, then there exists $\vec{u} \in \mathbb{Z}^2$ such that $(T^{\vec{u}}\eta)\restriction_F = f\restriction_F$, where by $\cdot\restriction_F$ we mean the restriction to the region F. So, for example, if η satisfies some complexity bound, such as the existence of a finite set $\mathcal{S} \subset \mathbb{Z}^2$ satisfying $P_\eta(\mathcal{S}) \leq N$ for some $N \geq 1$, then every $f \in X_\eta$ satisfies the same complexity bound. Moreover, if η is periodic with some period vector, then every $f \in X_\eta$ is also periodic with the same period vector. Similarly, if $\vec{u} \in \mathbb{Z}^2$ and $F \subset \mathbb{Z}^2$, there is a natural correspondence between a coloring of the form $(T^{-\vec{u}}f)\restriction_F$ and a coloring $f\restriction_{F+\vec{u}}$.

Characterizing periodicity of $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ amounts to studying properties of its orbit closure X_η. In particular, note that η is doubly periodic if and only if it has two non-commensurate period vectors, or equivalently X_η is finite.

1.4. Expansive and nonexpansive lines

Restricting a more general definition given by Boyle and Lind [2] to a dynamical system X with a continuous $\mathbb{Z}^2$-action $(T^{\vec{u}} : \vec{u} \in \mathbb{R}^2)$ on X, we say that a line $\ell \subset \mathbb{R}^2$ is an *expansive line* if there exist $r > 0$ and $\delta > 0$ such that whenever $f, g \in X$ satisfy $d(T^{\vec{u}}f, T^{\vec{u}}g) < \delta$ for all $\vec{u} \in \mathbb{Z}^2$ with $d(\vec{u}, \ell) < r$, then $f = g$. Any line that is not expansive is called a *nonexpansive line*.