European Journal of Combinatorics 43 (2015) 55–67

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Dually weighted Stirling-type sequences

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ARTICLE INFO

Article history:
Received 6 January 2014
Accepted 12 June 2014
Available online 26 July 2014

ABSTRACT

We introduce a generalization of the Stirling numbers via symmetric functions involving two weight functions. The resulting extension unifies previously known Stirling-type sequences with known symmetric function forms, as well as other sequences such as the p, q-binomial coefficients. Recurrence relations, generating functions, orthogonality relations, convolution formulas, and determinants of certain matrices involving the obtained sequences are derived. We also give combinatorial interpretations of certain cases in terms of colored partitions and permutations.

1. Introduction

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For $n, k \in \mathbb{N}$ such that $n \geq k$, the q-analogues of n, $n!$, and $\binom{n}{k}$, respectively are defined by $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, with $[0]_q = 0$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and $[0]_q! = 1$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. Also, corresponding p, q-analogues have been defined as $[n]_{pq} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + q^{n-1}$, with $[0]_{pq} = 0$, $[n]_{pq}! = [n]_{pq}[n-1]_{pq} \cdots [1]_{pq}$, and $[0]_{pq}! = 1$, and $\begin{bmatrix} n \\ k \end{bmatrix}_{pq} = \frac{[n]_{pq}!}{[k]_{pq}![n-k]_{pq}!}$. We denote by $c(n, k)$ and $S(n, k)$ the Stirling number of the first kind and the Stirling number of the second kind, respectively, and their p, q-analogues by $c_{p,q}[n, k]$ and $S_{p,q}[n, k]$. A number of authors have shown that these sequences have analogous symmetric function forms (see for example [21,9,8]). Specifically, if $e_t(x_0, x_1, \ldots, x_r)$ denote the t-th elementary symmetric function and $h_t(x_0, x_1, \ldots, x_r)$ the homogeneous symmetric

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http://dx.doi.org/10.1016/j.ejc.2014.06.010
0195-6698/

function on the set $\{x_0, x_1, \ldots, x_r\}$, then

$$q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = e_{n-k}(1, q, q^2, \ldots, q^{n-1})$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = h_{n-k}(1, q, q^2, \ldots, q^k)$$
$$c(n, k) = e_{n-k}(0, 1, 2, \ldots, n-1)$$
$$c_{p,q}[n, k] = e_{n-k}([0]_{p,q}, [1]_{p,q}, [2]_{p,q}, \ldots, [n-1]_{p,q})$$
$$S(n, k) = h_{n-k}(0, 1, 2, \ldots, k)$$
$$S_{p,q}[n, k] = h_{n-k}([0]_{p,q}, [1]_{p,q}, [2]_{p,q}, \ldots, [k]_{p,q}).$$

Using $[n]_{pq} = p^{n-1}[n]_{q/p}$, and hence, $[n]_{pq}! = p^{\binom{n}{2}}[n]_{q/p}!$, we can show that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q} = p^{k(n-k)} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{q/p,1}$. Applying the symmetric function expression for the q-binomial coefficients yields

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = h_{n-k}(p^k, p^{k-1}q, p^{k-2}q^2, \ldots, q^k). \quad (1)$$

Indeed, Médicis and Leroux [9] have shown that it is efficient to use symmetric functions in unifying the treatment of many Stirling-type sequences. In particular, they showed that the $\mathcal{U}$-Stirling numbers given below reduce to many previously known generalizations of Stirling numbers for specific choices of the weight function w:

$$c^{\mathcal{U}}(n, k) = e_{n-k}(w_0, w_1, \ldots, w_{n-1})$$
$$S^{\mathcal{U}}(n, k) = h_{n-k}(w_0, w_1, \ldots, w_k).$$

Nonetheless, there is a continued interest in other types of Stirling numbers. We mention here two recent generalizations of Stirling numbers that are also special cases of $\mathcal{U}$-Stirling numbers: Andrews and Littlejohn's Legendre Stirling numbers [1], obtained when $w_i = i(i+1)\cdots(i+m)$, and Miceli's poly-Stirling numbers [31], obtained by letting $w_i = p(i)$, where p is a polynomial in i with coefficients from $\mathbb{N}$. Moreover, it has been shown [26] that the p, q-binomial coefficients describe the magnetization distributions of the Ising model. Generalized Stirling numbers also occur as coefficients in the normal ordering of some non-commutative operators studied in mathematical physics (see [29] and references therein). It can be verified that these generalized Stirling numbers are equivalent to those that appear in transformations between certain generalized factorials [19,20].

In this paper, we introduce a generalization of the Stirling numbers which we shall call $\mathcal{V}$-Stirling numbers. They are inspired by the symmetric function form of the p, q-binomial coefficients in (1). Sections 3–6 deal with the recurrence relations, generating functions, orthogonality relations, and convolution formulas of these numbers. We also provide combinatorial interpretations of these numbers in Section 7 using certain colored permutations and partitions.

2. $\mathcal{V}$-Stirling numbers

Let $\mathcal{V} = (v, w)$, where v and w are weight functions from $\mathbb{Z}$ to a commutative ring K with unity, and let $\alpha, \beta \in \mathbb{Z}$. We define the *$\mathcal{V}$-Stirling numbers of the first kind and second kind*, respectively, as

$$c^{\mathcal{V}}_{\alpha,\beta}[n, k] := e_{n-k}(v_{\alpha+n-1}w_\beta, v_{\alpha+n-2}w_{\beta+1}, \ldots, v_\alpha w_{\beta+n-1}) \quad (2)$$

$$S^{\mathcal{V}}_{\alpha,\beta}[n, k] := h_{n-k}(v_{\alpha+k}w_\beta, v_{\alpha+k-1}w_{\beta+1}, \ldots, v_\alpha w_{\beta+k}) \quad (3)$$

for $n, k \in \mathbb{N}$ with $k \le n$. If $n, k < 0$, we set both $c^{\mathcal{V}}_{\alpha,\beta}[n, k]$ and $S^{\mathcal{V}}_{\alpha,\beta}[n, k]$ to be 0. Note that the parameters α and β do not give more generality than the weight functions themselves.

The $\mathcal{V}$-Stirling numbers reduce to the $\mathcal{U}$-Stirling numbers when $\mathcal{V} = (1, w)$ and $\alpha = \beta = 0$. We obtain the p, q-binomial coefficients when $v_i = p^i$ and $w_i = q^i$:

$$S^{\mathcal{V}}_{\alpha,\beta}[n, k] = p^{\alpha(n-k)} q^{\beta(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q}, \quad \text{and}$$