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Heterochromatic paths in edge colored graphs without small cycles and heterochromatic-triangle-free graphs

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ABSTRACT

Conditions for the existence of heterochromatic Hamiltonian paths and cycles in edge colored graphs are well investigated in literature. A related problem in this domain is to obtain good lower bounds for the length of a maximum heterochromatic path in an edge colored graph G. This problem is also well explored by now and the lower bounds are often specified as functions of the minimum color degree of G – the minimum number of distinct colors occurring at edges incident to any vertex of G – denoted by $\vartheta(G)$.

Initially, it was conjectured that the lower bound for the length of a maximum heterochromatic path for an edge colored graph G would be $\left\lceil \frac{2\vartheta(G)}{3} \right\rceil$. Chen and Li (2005) showed that the length of a maximum heterochromatic path in an edge colored graph G is at least $\vartheta(G) - 1$, if $1 \leq \vartheta(G) \leq 7$, and at least $\left\lceil \frac{3\vartheta(G)}{5} \right\rceil + 1$, if $\vartheta(G) \geq 8$. They conjectured that the tight lower bound would be $\vartheta(G) - 1$ and demonstrated some examples which achieve this bound. An unpublished manuscript from the same authors (Chen, Li) reported to show that if $\vartheta(G) \geq 8$, then G contains a heterochromatic path of length at least $\left\lceil \frac{2\vartheta(G)}{3} \right\rceil + 1$.

In this paper, we give lower bounds for the length of a maximum heterochromatic path in edge colored graphs without small cycles. We show that if G has no four cycles, then it contains a heterochromatic path of length at least $\vartheta(G) - o(\vartheta(G))$ and if the girth of G is at least $4\log_2(\vartheta(G)) + 2$, then it contains a heterochromatic

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path of length at least $\vartheta(G) - 2$, which is only one less than the bound conjectured by Chen and Li (2005). Other special cases considered include lower bounds for the length of a maximum heterochromatic path in edge colored bipartite graphs and triangle-free graphs: for triangle-free graphs we obtain a lower bound of $\left\lfloor \frac{5\vartheta(G)}{6} \right\rfloor$ and for bipartite graphs we obtain a lower bound of $\left\lceil \frac{6\vartheta(G)-3}{7} \right\rceil$.

In this paper, it is also shown that if the coloring is such that G has no heterochromatic triangles, then G contains a heterochromatic path of length at least $\left\lfloor \frac{13\vartheta(G)}{17} \right\rfloor$. This improves the previously known $\left\lceil \frac{3\vartheta(G)}{4} \right\rceil$ bound obtained by Chen and Li (2011). We also give a relatively shorter and simpler proof showing that any edge colored graph G contains a heterochromatic path of length at least $\left\lceil \frac{2\vartheta(G)}{3} \right\rceil$.

1. Introduction

An edge coloring of a graph is a mapping from its edge set to the set of natural numbers. If a graph G has an edge coloring specified, we call G an edge colored graph. The length of a path P is the number of edges of the path P. Unless specified otherwise, our graphs are finite simple graphs.

Let $G(V, E)$ be an edge colored graph. We use color(e) to denote the color given to an edge $e \in E$. (To denote the color given to an edge $(u, v) \in E$, we abuse the above notation and write color(u, v).) A heterochromatic or a rainbow subgraph in G is a subgraph H of G such that for every pair of distinct edges e_1 and e_2 of H, we have color$(e_1) \neq$ color(e_2).

The conditions for the existence of large heterochromatic subgraphs in edge colored graphs are well studied in literature [14,12,13,15]. Erdos et al. [10], Hahn et al. [14] and Albert et al. [1] gave some sufficient conditions on the coloring to guarantee a heterochromatic Hamiltonian cycle in an edge colored complete graph K_n. The conditions for the existence of heterochromatic Hamiltonian paths in infinite complete graphs were studied by Hahn and Thomassen [14] and later by Erdos and Tuza [11].

The number of distinct colors occurring at edges incident at a vertex v of G is called the color degree of v and is denoted by $deg^c(v)$. We use $\vartheta(G)$ to denote the minimum color degree of G, i.e., $\vartheta(G) = \min_{v \in V(G)} deg^c(v)$. Broersma et al. [3] obtained lower bounds for the length of a maximum heterochromatic path in an edge colored graph, in terms of its minimum color degree and minimum neighborhood union conditions. We use $\lambda(G)$ to denote the length of a maximum length heterochromatic path in G. They showed that for every vertex v of G, there exists a heterochromatic path starting at v and of length at least $\left\lceil \frac{\vartheta(G)+1}{2} \right\rceil$. They also showed that if for every pair of vertices x and y of G, the cardinality of the union of the colors given to edges incident with x and y is at least s, then $\lambda(G) \geq \left\lceil \frac{s}{3} \right\rceil + 1$.

Chen and Li [4] reported A. Saito's conjecture that $\lambda(G) \geq \left\lceil \frac{2\vartheta(G)}{3} \right\rceil$ for any edge colored graph G. They showed that $\lambda(G) \geq \vartheta(G) - 1$, if $3 \leq \vartheta(G) \leq 7$, and $\lambda(G) \geq \left\lceil \frac{3\vartheta(G)}{5} \right\rceil + 1$, if $\vartheta(G) \geq 8$. It is easy to see that if $\vartheta(G) = 1$ or 2, then $\lambda(G) \geq \vartheta(G)$. In the same paper, they conjectured that the actual bound could be $\vartheta(G) - 1$ and demonstrated some examples which achieve this bound. Recently, Das et al. [8] gave a simpler and shorter proof showing that $\lambda(G) \geq \left\lceil \frac{3\vartheta(G)}{5} \right\rceil$ for any edge colored graph G.

In an unpublished manuscript from Chen and Li [5], it was shown that if $\vartheta(G) \geq 8$, then $\lambda(G) \geq \left\lceil \frac{2\vartheta(G)}{3} \right\rceil + 1$. Further, in another work [6], they showed that if for every pair of vertices x and y of