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Ramsey numbers for bipartite graphs with small bandwidth[☆]

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ABSTRACT

We estimate Ramsey numbers for bipartite graphs with small bandwidth and bounded maximum degree. In particular we determine asymptotically the two and three color Ramsey numbers for grid graphs. More generally, we determine asymptotically the two color Ramsey number for bipartite graphs with small bandwidth and bounded maximum degree and the three color Ramsey number for such graphs with the additional assumption that the bipartite graph is balanced.

1. Introduction

For graphs $G_1, G_2, \dots, G_r$, the Ramsey number $R(G_1, G_2, \dots, G_r)$ is the smallest positive integer n such that if the edges of a complete graph K_n are partitioned into r disjoint color classes giving r graphs $H_1, H_2, \dots, H_r$, then at least one H_i $(1 \le i \le r)$ contains a subgraph isomorphic to G_i. The existence of such a positive integer follows from Ramsey's theorem. The number $R(G_1, G_2, \dots, G_r)$

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is called the Ramsey number of the graphs $G_1, G_2, \ldots, G_r$. Determining $R(G_1, G_2, \ldots, G_r)$ for general graphs appears to be a difficult problem (see e.g. [9] or [19]). For $r = 2$, a well-known theorem of Gerencsér and Gyárfás [8] states that

$$R(P_n, P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor,$$

where P_n denotes the path with $n \geq 2$ vertices. In [13] more general trees were considered. For a tree T, we write t_1 and t_2, with $t_2 \geq t_1$, for the sizes of the vertex classes of T as a bipartite graph. Note that $R(T, T) \geq 2t_1 + t_2 - 1$, since the following edge-coloring of $K_{2t_1+t_2-2}$ has no monochromatic copy of T. Partition the vertices into two classes V_1 and V_2 such that $|V_1| = t_1 - 1$ and $|V_2| = t_1 + t_2 - 1$, then use color "red" for all edges inside the two classes and use color "blue" for all edges between the classes. Furthermore, a similar edge-coloring of K_{2t_2-2} with two classes both of size $t_2 - 1$ shows that $R(T, T) \geq 2t_2 - 1$. Thus

$$R(T, T) \geq \max\{2t_1 + t_2, 2t_2\} - 1. \tag{1}$$

Haxell, Łuczak and Tingley provided in [13] an asymptotically matching upper bound for trees T with $\Delta(T) = o(t_2)$.

We partially extend this to bipartite graphs with small bandwidth and a more restrictive maximum degree condition. A graph $H = (W, E_H)$ is said to have *bandwidth* at most b, if there exists a labeling of the vertices by numbers $1, \ldots, n$ such that for every edge $ij \in E_H$ we have $|i - j| \leq b$. We focus our attention on the following class of bipartite graphs.

Definition 1.1. A bipartite graph H is called a (β, Δ)-graph if it has bandwidth at most $\beta|V(H)|$ and maximum degree at most Δ. Furthermore, we say that H is a *balanced* (β, Δ)*-graph* if it has a proper 2-coloring $\chi : V(H) \to [2]$ such that $\big|\, |\chi^{-1}(1)| - |\chi^{-1}(2)| \,\big| \leq \beta|\chi^{-1}(2)|$.

For example, it was shown in [5] that sufficiently large planar graphs with maximum degree at most Δ are (β, Δ)-graphs for any fixed $\beta > 0$. Our first theorem is an analogue of the result in [13] for (β, Δ)-graphs.

Theorem 1.2. *For every $\gamma > 0$ and natural number Δ, there exist a constant $\beta > 0$ and natural number n_0 such that for every (β, Δ)-graph H on $n \geq n_0$ vertices with a proper 2-coloring $\chi : V(H) \to [2]$ where $t_1 = |\chi^{-1}(1)|$ and $t_2 = |\chi^{-1}(2)|$, with $t_1 \leq t_2$, we have*

$$R(H, H) \leq (1 + \gamma) \max\{2t_1 + t_2, 2t_2\}.$$

For more recent results on the Ramsey number of graphs of higher chromatic number and sublinear bandwidth, we refer the reader to the work of Allen, Brightwell and Skokan [1].

For $r \geq 3$ colors less is known about Ramsey numbers. Let T be a tree and consider t_1 and t_2, with $t_1 \leq t_2$, the sizes of the vertex classes of T as a bipartite graph. For $r = 3$ colors the following construction gives a lower bound for $R(T, T, T)$. Partition the vertices of $K_{t_1+3t_2-3}$ into four classes, one special class V_0 with $|V_0| = t_1$ and three classes V_1, V_2 and V_3 of size $t_2 - 1$. The color for edges inside V_0 is arbitrary. Use color i inside the classes V_i and color i between V_i and V_0 for $1 \leq i \leq 3$. Finally, use color $k \in [3] \setminus \{i, j\}$ for edges between the classes V_i and V_j for $1 \leq i < j \leq 3$. It is easy to check that this coloring yields no monochromatic copy of T. Thus

$$R(T, T, T) \geq t_1 + 3t_2 - 2. \tag{2}$$

Proving a conjecture of Faudree and Schelp [6], it was shown in [10] that this construction is optimal for large paths, i.e., for sufficiently large n we have

$$R(P_n, P_n, P_n) = \begin{cases} 2n - 1 & \text{for odd } n, \\ 2n - 2 & \text{for even } n. \end{cases}$$

Asymptotically this was also proved independently by Figaj and Luczak [7]. Benevides and Skokan [2] proved that $R(C_n, C_n, C_n) = 2n$ for sufficiently large even n. Our second result extends the two above ones asymptotically to balanced (β, Δ)-graphs.