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Enumeration and classification of self-orthogonal partial Latin rectangles by using the polynomial method

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ABSTRACT

The current paper deals with the enumeration and classification of the set $\mathcal{SOR}_{r,n}$ of self-orthogonal $r \times r$ partial Latin rectangles based on n symbols. These combinatorial objects are identified with the independent sets of a Hamming graph and with the zeros of a radical zero-dimensional ideal of polynomials, whose reduced Gröbner basis and Hilbert series can be computed to determine explicitly the set $\mathcal{SOR}_{r,n}$. In particular, the cardinality of this set is shown for $r \leq 4$ and $n \leq 9$ and several formulas on the cardinality of $\mathcal{SOR}_{r,n}$ are exposed, for $r \leq 3$. The distribution of $r \times s$ partial Latin rectangles based on n symbols according to their size is also obtained, for all $r, s, n \leq 4$.

1. Introduction

An $r \times s$ *partial Latin rectangle* based on $[n] = \{1, \ldots, n\}$ is an $r \times s$ array in which each cell is either empty or contains a symbol of $[n]$, such that each symbol occurs at most once in each row and in each column. Its number of filled cells is its *size*. Let $\mathcal{R}_{r,s,n}$ and $\mathcal{R}_{r,s,n:m}$ respectively denote the set of $r \times s$ partial Latin rectangles based on $[n]$ and its subset of partial Latin rectangles of size m. Given $P = (p_{ij}) \in \mathcal{R}_{r,s,n}$, its *orthogonal array representation* is the set $O(P) = \{(i, j, p_{ij}) : i \in [r], j \in [s], p_{ij} \in [n]\}$. Permutations of rows, columns and symbols of P give rise to new $r \times s$ partial Latin rectangles based on $[n]$, which are said to be *isotopic* to P. If S_m denotes the symmetric group on m

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1	3		
2		3	1
	1	2	
	2		3

Fig. 1. Example of a self-orthogonal 4×4 partial Latin rectangle based on [3].

elements, then $\Theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ is an *isotopism* of $\mathcal{R}_{r,s,n}$ and it is defined the isotopic partial Latin rectangle P^{Θ} such that $O(P^{\Theta}) = \{(\alpha(i), \beta(j), \gamma(p_{i,j})) : i \in [r], j \in [s], p_{ij} \in [n]\}$. Given a permutation $\pi \in S_3$, it is defined the *parastrophic* partial Latin rectangle P^{π} such that $O(P^{\pi}) = \{(p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) : (p_1, p_2, p_3) \in O(P)\}$. If $P^{\pi} \in \mathcal{R}_{r,s,n}$, then π is called a *parastrophism* of $\mathcal{R}_{r,s,n}$. The composition of an isotopism and a parastrophism is a *paratopism*. Two partial Latin rectangles are in the same *main class* if one of them is isotopic to a parastrophic partial Latin rectangle of the other.

Two partial Latin rectangles $P = (p_{ij}), Q = (q_{ij}) \in \mathcal{R}_{r,s,n}$ are *orthogonal* if, given $i, i' \in [r]$ and $j, j' \in [s]$ such that $p_{ij} = p_{i'j'} \in [n]$, then q_{ij} and $q_{i'j'}$ are not the same symbol of $[n]$. If $r = s$, then the partial Latin rectangle P is *self-orthogonal* if it is orthogonal to its transpose P^t (see Fig. 1).

Let $\mathcal{SOR}_{r,n}$ be the set of self-orthogonal $r \times r$ partial Latin rectangles based on $[n]$. Only those isotopisms of the form $(\alpha, \alpha, \gamma) \in S_r \times S_r \times S_n$ and those paratopisms based on $\overline{S}_3 = \{(1)(2)(3), (12)(3)\}$ preserve always the set $\mathcal{SOR}_{r,n}$. Hence, the sets $S_r \times S_n$ and $S_r \times S_n \rtimes \overline{S}_3$ determine, respectively, the *isotopism* and *paratopism groups* of $\mathcal{SOR}_{r,n}$. The enumeration of isotopism and main classes of $\mathcal{SOR}_{r,n}$ has been studied for $r = n \leq 10$ [5,6,13]. However, there does not exist a similar study for self-orthogonal partial Latin rectangles of any order. In the current paper, we deal with this problem by adapting the *Combinatorial Nullstellensatz* of Alon [1], whose effectiveness in the study of Latin squares has been exposed in [12,11].

The paper is organized as follows. In Section 2, we indicate some preliminaries concepts and results on commutative algebra. In Section 3, the set $\mathcal{R}_{r,s,n}$ is identified with that of independent sets of a Hamming graph and with the set of zeros of a zero-dimensional radical ideal, whose reduced Gröbner bases and Hilbert series determine, respectively, the elements and cardinality of $\mathcal{R}_{r,s,n:m}$, for all natural m. This cardinality is explicitly shown for $r \leq s \leq n \leq 4$ and $m \leq rsn$. In Section 4, we consider new polynomials to be added to the above ideal in order to determine the set $\mathcal{SOR}_{r,n}$. Besides, two strategies are indicated that allow us to reduce the cost of computation of the Gröbner basis and Hilbert series of the new ideal. They are used to determine the cardinality of $\mathcal{SOR}_{r,n}$ for $r \leq 4$ and $n \leq 9$. Some general formulas about the cardinality of $\mathcal{SOR}_{r,n}$ are finally exposed, for $r \leq 3$.

2. Preliminaries

We start with some basic concepts of commutative algebra (see [7,8,15] for more details). Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field k with the *standard grading* induced by the degree of polynomials, that is, $R = \bigoplus_{0 \leq d} R_d$, where each R_d is the set of homogeneous polynomials in R of degree d. The largest monomial of a polynomial of R with respect to a given term order $<$ is its *initial monomial*. Given an ideal I of R, the ideal generated by the initial monomials with respect to $<$ of all the non-zero elements of I is its *initial ideal* $I_<$. Any monomial of R which is not contained in $I_<$ is called a *standard monomial* of I with respect to $<$. The set of standard monomials of I with respect to any given term order can be used to study the dimension of the quotient ring R/I. This ring inherits the natural grading of R and can be written as the direct sum $\bigoplus_{0 \leq d} R_d/I_d$, where $I_d = R_d \cap I$. In particular, the set of standard monomials of I of degree d with respect to a given term order constitutes a linear k-basis of R_d/I_d and hence, its cardinality coincides with $\dim_k(R_d/I_d)$, regardless of the term order which has been chosen. The *Hilbert function* $\mathrm{HF}_{R/I}$ of R/I maps each non-negative integer d onto $\dim_k(R_d/I_d)$. Its *Hilbert series* is the generating function $\mathrm{HS}_{R/I}(t) = \sum_{0 \leq d} \mathrm{HF}_{R/I}(d) \cdot t^d$, which can also be written as:

$$\mathrm{HS}_{R/I}(t) = \frac{P(t)}{(1-t)^n} = \frac{Q(t)}{(1-t)^{\dim_k(I)}}, \tag{1}$$