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On the existence of radial Moore graphs for every radius and every degree

José Gómez [a,1], Mirka Miller [b,c]

[a] Departament de Matematica Aplicada IV, Universitat Politècnica de Catalunya, Barcelona, Spain

[b] School of Mathematical and Physical Sciences, University of Newcastle, Australia

[c] Department of Mathematics, University of West Bohemia, Pilsen, Czech Republic

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ABSTRACT

The degree/diameter problem is to determine the largest graphs of given maximum degree and given diameter. General upper bounds – called Moore bounds – for the order of such graphs are attainable only for certain special graphs, called Moore graphs. Moore graphs are scarce and so the next challenge is to find graphs which are somehow "close" to the nonexistent ideal of a Moore graph by holding fixed two of the parameters, order, diameter and maximum degree, and optimising the third parameter. In this paper we consider the existence of graphs that have order equal to Moore bound for given radius and maximum degree and as the relaxation we require the diameter to be at most one more than the radius. Such graphs are called radial Moore graphs. In this paper we prove that radial Moore graphs exist for every diameter and every sufficiently large degree, depending on the diameter.

1. Introduction

The topology of a network (such as a telecommunications, multiprocessor, or local area network, to name just a few) is usually modelled by a graph in which vertices represent 'nodes' (stations or processors) while edges stand for 'links' or other types of connections.

E-mail addresses: jgomez@ma4.upc.edu (J. Gómez), mirka.miller@newcastle.edu.au (M. Miller).

[1] Deceased author.

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In the design of such networks, there are a number of features that must be taken into account. The most common ones, however, seem to be limitations on the vertex degrees and on the diameter. The network interpretation of these two parameters is obvious: The degree of a vertex is the number of the connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes. What is then the largest number of nodes in a network with a limited degree and diameter? This question leads to the

- *Degree/Diameter Problem*: Given natural numbers Δ and D, find the largest possible number of vertices $n_{\Delta,D}$ in a graph of maximum degree Δ and diameter D.

There is a straightforward upper bound on the largest possible order (i.e., the number of vertices) $n_{\Delta,D}$ of a graph G of maximum degree Δ and diameter D. Trivially, if $\Delta = 1$ then $D = 1$ and $n_{1,1} = 2$; therefore, in what follows we shall assume that $\Delta \geq 2$.

Let v be a vertex of the graph G and let n_i, for $0 \leq i \leq D$, be the number of vertices at distance i from v. Since a vertex at distance $i \geq 1$ from v can be adjacent to at most $\Delta - 1$ vertices at distance $i+1$ from v, we have $n_{i+1} \leq (\Delta - 1)n_i$, for all i such that $1 \leq i \leq D-1$. Since $n_1 \leq \Delta$, it follows that $n_i \leq \Delta(\Delta-1)^{i-1}$, for $1 \leq i \leq D$ and so

$$\begin{aligned} n_{\Delta,D} = \sum_{i=0}^{D} n_i &\leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta(1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}) \\ &= \begin{cases} 1 + \Delta \dfrac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2. \end{cases} \end{aligned} \tag{1}$$

The right-hand side of (1) is called the *Moore bound* and is denoted by $M_{\Delta,D}$ [6]. A graph whose order is equal to the Moore bound $M_{\Delta,D}$ is called a *Moore graph*. It is easy to prove that such a graph is necessarily regular of degree Δ.

The study of Moore graphs was initiated by Hoffman and Singleton. Their paper [6] treated Moore graphs of diameter 2 and 3. In the case of diameter $D = 2$, they proved that Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57 but for no other degree, and that for $\Delta = 2, 3, 7$ the graphs are unique. For $D = 3$ they showed that the unique Moore graph is the heptagon (for $\Delta = 2$). No Moore graphs exist for the parameters $\Delta \geq 3$ and $D \geq 3$. This was proved by Damerell [4] and independently also by Bannai and Ito [2].

The known Moore graphs are: for diameter $D = 1$ and degree $\Delta \geq 1$ the complete graphs $K_{\Delta+1}$; for degree $\Delta = 2$, the cycles C_{2D+1}. Furthermore, for diameter $D = 2$, there is also the Petersen graph for degree $\Delta = 3$, and the Hoffman–Singleton graph for degree $\Delta = 7$. For diameter $D \geq 3$ and degree $\Delta = 2$, there are no Moore graphs other than the cycles on $2D + 1$ vertices. Interestingly, despite many efforts by many researchers, there still remains one unknown case: it is not known whether a Moore graph of diameter 2 and degree 57 exists or not.

Since, apart from the complete graphs and odd cycles, Moore graphs exist for only a few combinations of the degree and diameter values, we are interested in studying the existence of large graphs which are in some way 'close' to Moore graphs. Given that we are dealing with three parameters, namely, order, degree and diameter, in order to get close to Moore graphs, we may consider relaxing each of these parameters in turn. In fact, the degree/diameter problem can be seen as a relaxation, for a given diameter D and maximum degree Δ, of the order of a graph to being as close as possible to the Moore bound $M_{\Delta,D}$, approaching the bound from below. An initial discussion on the relaxation of the three parameters was presented by Miller and Pineda-Villavicencio in [9].

Relaxing the order: Relaxing the order to $M_{\Delta,D} - \delta$, with the *defect* δ, corresponds to the degree/diameter problem. For more details on the degree/diameter problem, see the survey [10].

Relaxing the degree: As the maximum degree is a global measure of the degrees of the vertices of a graph, we could choose a finer measure, for example, the degree sequence. This approach could be dealt with in several ways. A graph could be considered to be close to a Moore graph if it has $M_{\Delta,D}$ vertices, diameter D and if, for example,