European Journal of Combinatorics 47 (2015) 75–94

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Edge-colorings avoiding a fixed matching with a prescribed color pattern

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ARTICLE INFO

Article history:
Received 12 May 2014
Accepted 22 January 2015
Available online 16 February 2015

ABSTRACT

We consider an extremal problem motivated by a question of Erdős and Rothschild (Erdős, 1974) regarding edge-colorings of graphs avoiding a given monochromatic subgraph. An extension of this problem to edge-colorings avoiding fixed subgraphs with a prescribed coloring has been studied by Balogh (Balogh, 2006). In this work, we consider the following natural generalization of the original Erdős–Rothschild question: given a natural number r and a graph F, an r-pattern P of F is a partition of the edge set of F into r (possibly empty) classes, and an r-coloring of the edge set of a graph G is said to be (F, P)-free if it does not contain a copy of F in which the partition of the edge set induced by the coloring has a copy of P. Let $c_{r,(F,P)}(G)$ be the number of (F, P)-free r-colorings of a graph G. For large n, we maximize this number over all n-vertex graphs for a large class of patterns in matchings and we describe the graphs that achieve this maximum.

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1. Introduction

One of the most natural extremal problems in graph theory consists in determining the largest graphs with a prescribed structure among all graphs with a given number of vertices. For instance, a simple problem in this direction is to decide, among all triangle-free n-vertex graphs, which is

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http://dx.doi.org/10.1016/j.ejc.2015.01.011
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the graph with the maximum number of edges. This question was answered by Mantel [11] at the beginning of the twentieth century, giving rise to an active branch of research. More generally, for any fixed graph F, we say that a graph G is F-free if it does not contain F as a subgraph. Finding the maximum number of edges among all F-free n-vertex graphs, and determining the class of n-vertex graphs that achieve this number is now known as the *Turán problem* associated with F, which was solved for complete graphs in [14]. The maximum number of edges in an F-free n-vertex graph is denoted by $\text{ex}(n, F)$ and the n-vertex graphs that achieve this bound are called *F-extremal*. This is one of the most popular problems in extremal graph theory and there is a vast literature related with it (for more information and recent developments, we refer to Keevash [8], and the references therein).

In connection with a question of Erdős and Rothschild [3], several authors have investigated the following related problem. Instead of looking for F-free n-vertex graphs, they were interested in *edge colorings* of graphs on n vertices such that *every color class is F-free*. (We observe that edge colorings in this work are not necessarily proper.) More precisely, given an integer $r \geq 1$ and a graph F containing at least one edge, one considers the function that associates, with a graph G, the number $c_{r,F}(G)$ of r-colorings of the edge set of G for which there is no monochromatic copy of F. The problem consists of finding $c_{r,F}(n)$, the maximum of $c_{r,F}(G)$ over all n-vertex graphs G. For instance, if there is a single color available, we must have $c_{1,F}(n) = 1$, with equality $c_{1,F}(n) = c_{1,F}(G)$ for every graph G on n vertices that does not contain a copy of F. For simplicity, we assume that the colors lie in the set $\{1, \ldots, r\}$. The main motivation for considering this function is its connection with $\text{ex}(n, F)$:

$$c_{r,F}(n) \geq r^{\text{ex}(n,F)} \quad \text{for every } n \geq 2. \tag{1}$$

This holds trivially, as any r-coloring of the edges of an F-extremal n-vertex graph is F-free, and there are precisely $r^{\text{ex}(n,F)}$ such colorings. Moreover, with the Regularity Lemma [13], one can show that, for $r \in \{2, 3\}$ and n sufficiently large, we have

$$c_{r,F}(n) \leq r^{\text{ex}(n,F)+o(n^2)}$$

for graphs F with $\text{ex}(n, F) = \Theta(n^2)$.

Erdős and Rothschild were interested in instances for which (1) is tight, and they conjectured that this was the case for $r = 2$ and $F = K_3$, which was proved by Yuster [15]. Since then, the function $c_{r,F}(n)$ has been studied for several instances of graphs, such as complete graphs [1,12,15], odd cycles [1], matchings [6], paths and stars [7]. The following turns out to be a common feature of several instances of F, such as complete graphs, odd cycles and matchings (but not of paths and stars): when the number of colors is either two or three, inequality (1) holds with equality for n sufficiently large. Moreover, the class of *(r, F)-extremal* configurations, that is, the class of n-vertex graphs G such that $c_{r,F}(G) = c_{r,F}(n)$, is equal to the class of F-extremal graphs if n is sufficiently large; however, the F-extremal graphs are not (r, F)-extremal when at least four colors are used.

More recently, Balogh [2] has considered a variant of this problem. For a fixed graph F, he studied r-colorings of the edge set of a graph G that do not contain a copy of F colored *according to a fixed coloring*. For instance, if $r = 3$, $F = K_3$ is a triangle and $\hat{F}$ is a coloring of K_3 in which each of the three colors appears exactly once, we are looking for 3-colorings of the edges with no *rainbow* triangles. Let $c_{r,\hat{F}}(n)$ be the maximum number of r-colorings of an n-vertex graph with no copy of F colored as $\hat{F}$. Balogh has proved that, for any coloring $\hat{F}$ of the complete graph $F = K_\ell$ with two colors, we have $c_{2,\hat{F}}(n) = 2^{\text{ex}(n,K_\ell)}$ for n sufficiently large. However, this picture changes if we consider the previous example of 3-colorings with no rainbow triangles: Balogh noticed that, if we color the complete graph K_n with any two of the three colors available, there is no rainbow copy of K_3, which gives at least $3 \cdot 2^{\binom{n}{2}} - 3 \gg 3^{\text{ex}(n,K_3)} = 3^{n^2/4+o(n^2)}$ distinct (K_3, P)-free colorings. (As usual, we say that two positive functions g, f satisfy $g(n) \ll f(n)$ if $\lim_{n\to\infty} g(n)/f(n) = 0$.)

In this paper we consider a related problem, which deals with r-colorings of the edge set of a graph G that do not contain a copy of F colored *according to a fixed pattern*. An *r-pattern* P of a graph F is a partition of the edge set of F into r (possibly empty) classes, and an r-coloring of the edge set of G is said to be *(F, P)-free* if it does not contain a copy of F in which the partition of the edge set induced by the coloring has a copy of P. Again, given a graph G, one may consider the number $c_{r,(F,P)}(G)$ of (F, P)-free r-colorings of G, while $c_{r,(F,P)}(n)$ is defined as the maximum of this quantity over all

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