European Journal of Combinatorics 47 (2015) 115–122

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

A note on the shameful conjecture

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ARTICLE INFO

Article history:
Received 19 August 2014
Accepted 1 February 2015
Available online 27 February 2015

ABSTRACT

Let $P_G(q)$ denote the chromatic polynomial of a graph G on n vertices. The 'shameful conjecture' due to Bartels and Welsh states that,

$$\frac{P_G(n)}{P_G(n-1)} \geq \frac{n^n}{(n-1)^n}.$$

Let $\mu(G)$ denote the expected number of colors used in a uniformly random proper n-coloring of G. The above inequality can be interpreted as saying that $\mu(G) \geq \mu(O_n)$, where O_n is the empty graph on n nodes. This conjecture was proved by F.M. Dong, who in fact showed that,

$$\frac{P_G(q)}{P_G(q-1)} \geq \frac{q^n}{(q-1)^n}$$

for all $q \geq n$. There are examples showing that this inequality is not true for all $q \geq 2$. In this paper, we show that the above inequality holds for all $q \geq 36D^{3/2}$, where D is the largest degree of G. It is also shown that the above inequality holds true for all $q \geq 2$ when G is a claw-free graph.

1. Introduction

The chromatic polynomial is an important algebraic object studied in the field of graph coloring. For a graph $G = (V, E)$ with vertex set V and edge set E, let $\sigma : V \to \{1, \ldots, q\}$ be a map. The map σ is said to be a proper q-coloring of graph G if for every edge in E the endpoints of the edge have

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http://dx.doi.org/10.1016/j.ejc.2015.02.001
0195-6698/

distinct images under σ. Let $P_G(q)$ denote the total number of proper q-colorings of G. It is well known that $P_G(q)$ is a polynomial in q and is known as the chromatic polynomial. In fact,

$$P_G(q) = \sum_{E' \subseteq E} (-1)^{|E'|} q^{C(E')}, \tag{1}$$

where $C(E')$ denotes the number of connected components of (V, E'). The sum goes over all subsets $E' \subseteq E$ of the edge set E. This is easily seen using the inclusion–exclusion principle as is explained in Section 2.

Properties of the chromatic polynomial have been studied extensively. For example, the log-concavity of the chromatic polynomial (proved for its coefficients [14]) is well studied [4,3,17]. There has also been a lot of interest in understanding the roots of the chromatic polynomial [6,5,16,2,13].

Bartels and Welsh [1] studied a Markov chain on colorings, which would help approximate $\mu(G)$, the expected number of colors in a uniformly random proper n-coloring on G. They proved that,

$$\mu(G) = n\left(1 - \frac{P_G(n-1)}{P_G(n)}\right), \tag{2}$$

where $n = |V|$.

They conjectured that on the set of graphs on n nodes, μ is minimized when G is the empty graph, O_n, on n vertices, that is,

$$n\left(1 - \frac{P_G(n-1)}{P_G(n)}\right) = \mu(G) \geq \mu(O_n) = n\left(1 - \left(\frac{n-1}{n}\right)^n\right). \tag{3}$$

Rewriting this, the conjecture states:

$$\frac{P_G(n)}{P_G(n-1)} \geq \frac{n^n}{(n-1)^n}. \tag{4}$$

They point out that the inequality

$$\frac{P_G(q)}{P_G(q-1)} \geq \frac{q^n}{(q-1)^n} \tag{5}$$

may not hold true for all $q \geq 2$, as was shown by the following example due to Colin McDiarmid. Let $G = K_{n,n}$, the complete bipartite graph with partitions of size n each with $n \geq 10$. Then,

$$\frac{P_G(3)}{P_G(2)} = \frac{6 + 6(2^n - 2)}{2} = \frac{3(2^n - 1)}{1} < \frac{3^{2n}}{2^{2n}}. \tag{6}$$

This conjecture came to be dubbed as the 'shameful conjecture' and was proved 5 years later by Dong [12]. In fact, Dong proved the stronger result that inequality (5) holds true for all $q \geq n - 1$. In particular, this implies,

$$\frac{P_G(n)}{P_G(n-1)} \geq \frac{n^n}{(n-1)^n} \geq e. \tag{7}$$

Before that Seymour [15] also showed that,

$$\frac{P_G(n)}{P_G(n-1)} \geq \frac{685}{252} = 2.7182539\ldots < e. \tag{8}$$