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More on quasi-random graphs, subgraph counts and graph limits

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ABSTRACT

We study some properties of graphs (or, rather, graph sequences) defined by demanding that the number of subgraphs of a given type, with vertices in subsets of given sizes, approximatively equals the number expected in a random graph. It has been shown by several authors that several such conditions are quasi-random, but that there are exceptions. In order to understand this better, we investigate some new properties of this type. We show that these properties too are quasi-random, at least in some cases; however, there are also cases that are left as open problems, and we discuss why the proofs fail in these cases.

The proofs are based on the theory of graph limits; and on the method and results developed by Janson (2011), this translates the combinatorial problem to an analytic problem, which then is translated to an algebraic problem.

1. Introduction

Consider a sequence of graphs (G_n), with $|G_n| \to \infty$ as $n \to \infty$. Thomason [18,19] and Chung, Graham and Wilson [4] showed that a number of different 'random-like' properties of the sequence (G_n) are equivalent, and we say that (G_n) is *quasi-random*, or more precisely *p-quasi-random*, if it satisfies these properties. (Here $p \in [0, 1]$ is a parameter.) Many other equivalent properties of different types have later been added by various authors. We say that a property of sequences (G_n) of graphs

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(with $|G_n| \to \infty$) is a *quasi-random property* (or more specifically a *p-quasi-random property*) if it characterizes quasi-random (or p-quasi-random) sequences of graphs.

One of the quasi-random properties considered by Chung, Graham and Wilson [4] is based on subgraph counts, see (2.2). Further quasi-random properties based on restricted subgraph count properties have been found by Chung and Graham [3], Simonovits and Sós [15,17], Shapira [12], Shapira and Yuster [13,14], Yuster [20], Janson [7], Huang and Lee [6], see Section 2.

The purpose of the present paper is to continue the study of such properties by considering some further cases not treated earlier; in particular (Theorems 2.11 and 2.12), we prove that some further properties of this type are quasi-random. Our main purpose is not to just add to the already long list of quasi-random properties; we hope that this study will contribute to the understanding of this type of quasi-random property, and in particular explain why the case in Theorem 2.12 is more difficult than the one in Theorem 2.11. (See also Section 9 for a discussion of further similar properties.)

We use the method of Janson [7] based on graph limits. We assume that the reader is familiar with the basics of the theory of graph limits and graphons developed in e.g. Lovász and Szegedy [9] and Borgs, Chayes, Lovász, Sós and Vesztergombi [1]; otherwise, see Janson [7] (for the present context) or the comprehensive book by Lovász [8]. As is well-known, there is a simple characterization of quasi-random sequences in terms of graph limits: a sequence (G_n) with $|G_n| \to \infty$ is p-quasi-random if and only if $G_n \to W_p$, where W_p is the graphon that is constant with $W_p = p$ [1,2,9], see also [8, Section 1.4.2 and Example 11.37]. (Indeed, quasi-random graphs form one of the roots of graph limit theory.)

The idea of the method is to use this characterization to translate the property of graph sequences to a property of graphons (see Section 3), and then show that only constant graphons satisfy this property. It turns out that this leads to both analytic (Section 4) and algebraic (Section 6) problems, which we find interesting in themselves. We have only partly succeeded to solve these problems, so we leave several open problems.

Remark 1.1. Many of the references above use Szemerédi's regularity lemma as their main tool to study quasi-random properties, and it has been known since [16] that quasi-randomness can be characterized using Szemerédi partitions. It is also well-known that there are strong connections between Szemerédi's regularity lemma and graph limits, see [1,10,8], so on a deeper level the methods are related although they superficially look very different. (It thus might be possible to translate arguments of one type to the other, although it is far from clear how this might be done.) Both methods lead also to the same (sometimes difficult) algebraic problems. As discussed in [7], the method used here eliminates the many small error terms in the regularity lemma approach; on the other hand, it leads to analytic problems with no direct counterpart in the other approach. It is partly a matter of taste what type of arguments one prefers.

2. Notation, background and main results

All graphs in this paper are finite, undirected and simple. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$. We write $|G| := |V(G)|$ for the number of vertices of G, and $e(G) := |E(G)|$ for the number of edges. As usual, $[n] := \{1, \dots, n\}$.

All unspecified limits in this paper are as $n \to \infty$, and $o(1)$ denotes a quantity that tends to 0 as $n \to \infty$. We will often use $o(1)$ for quantities that depend on some subset(s) of a vertex set $V(G)$; we then always implicitly assume that the convergence is uniform for all choices of the subsets. We interpret $o(a_n)$ for a given sequence a_n similarly.

Let F and G be labelled graphs. For convenience, we assume throughout the paper (when it matters) that $V(F) = [|F|] = \{1, \dots, |F|\}$. We generally let $m = |F|$.

Definition 2.1. (i) $N(F, G)$ is the number of labelled copies of F in G (not necessarily induced); equivalently, $N(F, G)$ is the number of injective maps $\varphi : V(F) \to V(G)$ that are graph homomorphisms (i.e., if i and j are adjacent in F, then $\varphi(i)$ and $\varphi(j)$ are adjacent in G).

(ii) If $U_1, \dots, U_{|F|}$ are subsets of $V(G)$, let $N(F, G; U_1, \dots, U_{|F|})$ be the number of labelled copies of F in G with the ith vertex in U_i; equivalently, $N(F, G; U_1, \dots, U_{|F|})$ is the number of injective graph