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Binary matroids and local complementation

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ABSTRACT

We introduce a binary matroid $M[IAS(G)]$ associated with a looped simple graph G. $M[IAS(G)]$ classifies G up to local equivalence, and determines the delta-matroid and isotropic system associated with G. Moreover, a parametrized form of its Tutte polynomial yields the interlace polynomials of G.

1. Introduction

A *graph* $G = (V(G), E(G))$ consists of a finite vertex-set $V(G)$ and a finite edge-set $E(G)$. Each edge is incident on one or two vertices; an edge incident on only one vertex is a *loop*. The two vertices incident on a non-loop edge are *neighbors*, and the *open neighborhood* of a vertex v is $N(v) = \{\text{neighbors of } v\}$. A graph in which different edges can be distinguished by their vertex-incidences is a *looped simple graph*, and a *simple graph* is a looped simple graph with no loop.

In this paper we are concerned with properties of looped simple graphs motivated by two sets of ideas. The first set of ideas is the theory of the principal pivot transform (PPT) over $GF(2)$. PPT over arbitrary fields was introduced more than 50 years ago by Tucker [39]; see also the survey of Tsatsomeros [38]. According to Geelen [25], PPT transformations applied to the mod-2 adjacency matrices of looped simple graphs are generated by two kinds of *elementary* PPT operations, *non-simple local complementations* with respect to looped vertices and *edge pivots* with respect to edges connecting unlooped vertices. The second set of ideas is the theory of 4-regular graphs and their Euler circuits, initiated more than 40 years ago by Kotzig [28]. Kotzig proved that all the Euler circuits of a 4-regular graph are obtained from any one using κ*-transformations*. If a 4-regular graph is directed in such a way that every vertex has indegree 2 and outdegree 2, then Kotzig [28], Pevzner [30] and Ukkonen [40] showed that all of the graph's directed Euler circuits are obtained from any one through certain combinations of κ-transformations called *transpositions* by Arratia, Bollobás and Sorkin [2–4].

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Bouchet [8] and Rosenstiehl and Read [31] introduced a simple graph associated with any Euler circuit of a connected 4-regular graph, the *alternance* graph or *interlacement* graph; an equivalent *link relation matrix* was defined by Cohn and Lempel [21] in the context of the theory of permutations. These authors showed that the effects of κ-transformations and transpositions on interlacement graphs are given by *simple* local complementations and edge pivots, respectively.

In the late 1980s, Bouchet introduced two new kinds of combinatorial structures associated with these two theories. On the one hand are the *delta-matroids* [9], some of which are associated with looped simple graphs. The fundamental operation of delta-matroid theory is a way of changing one delta-matroid into another, called *twisting*. Two looped simple graphs are related through PPT operations if and only if their associated delta-matroids are related through twisting. On the other hand are the *isotropic systems* [10,12], all of which are associated with *fundamental graphs*. Two isotropic systems are *strongly isomorphic* if and only if they share fundamental graphs. Moreover, two simple graphs are related through simple local complementations if and only if they are fundamental graphs of strongly isomorphic isotropic systems. Properties of isotropic systems were featured in the proof of Bouchet's famous "forbidden minors" characterization of circle graphs [14].

The purpose of this paper is to introduce a binary matroid constructed in a natural way from the adjacency matrix of a looped simple graph G; we call it the *isotropic matroid of* G, in honor of Bouchet's isotropic systems. Let G be a looped simple graph with adjacency matrix $A(G)$. That is, $A(G)$ is the $|V(G)| \times |V(G)|$ matrix with entries in $GF(2)$ given by: a diagonal entry is 1 if and only if the corresponding vertex is looped, and an off-diagonal entry is 1 if and only if the corresponding vertices are adjacent. Let $IAS(G)$ denote the $|V(G)| \times (3\,|V(G)|)$ matrix

$$IAS(G) = (I \mid A(G) \mid I + A(G)).$$

Definition 1. The isotropic matroid of G is the binary matroid $M[IAS(G)]$ represented by $IAS(G)$.

Let $W(G)$ denote the ground set of $M[IAS(G)]$, i.e., the set of columns of $IAS(G)$. If $v \in V(G)$ then there are three columns of $IAS(G)$ corresponding to v: one in I, one in $A(G)$, and one in $I + A(G)$. For notational convenience, and to indicate the connection with our work on interlace polynomials [33,36,37], we use v_ϕ to denote the column of I corresponding to v, v_χ to denote the column of $A(G)$ corresponding to v, and v_ψ to denote the column of $I + A(G)$ corresponding to v. The set $\{v_\phi, v_\chi, v_\psi\}$ is the *vertex triple* corresponding to v.

Notice that if G_2 is obtained from G_1 by loop complementation at a vertex v then there is an isomorphism between the isotropic matroids $M[IAS(G_1)]$ and $M[IAS(G_2)]$ that simply interchanges the v_χ and v_ψ elements of $W(G_1)$ and $W(G_2)$. We say isomorphisms like this, which map vertex triples to vertex triples, are *compatible with the partitions of* $W(G_1)$ *and* $W(G_2)$ *into vertex triples*, or simply *compatible*. In Section 4 we observe that edge pivots and local complementations also induce compatible isomorphisms of isotropic matroids. Moreover, every compatible isomorphism is induced by some sequence of edge pivots, local complementations and loop complementations. It follows that compatible isomorphisms of isotropic matroids classify simple graphs and looped simple graphs under various combinations of these operations. For instance:

Theorem 2. *Let G_1 and G_2 be simple graphs. Then the following conditions are equivalent:*

1. *Up to isomorphism, G_2 can be obtained from G_1 using simple local complementations.*
2. *There is a compatible isomorphism $M[IAS(G_1)] \cong M[IAS(G_2)]$.*

Theorem 3. *Let G_1 and G_2 be looped simple graphs. Then the following conditions are equivalent:*

1. *Up to isomorphism, G_2 can be obtained from G_1 using local complementations and loop complementations.*
2. *There is a compatible isomorphism $M[IAS(G_1)] \cong M[IAS(G_2)]$.*

Theorem 4. *Let G_1 and G_2 be simple graphs. Then the following conditions are equivalent:*

1. *Up to isomorphism, G_2 can be obtained from G_1 using edge pivots.*
2. *There is a compatible isomorphism $M[IAS(G_1)] \cong M[IAS(G_2)]$, which maps ψ elements to ψ elements.*