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The local eigenvalues of a bipartite distance-regular graph

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ABSTRACT

We consider a bipartite distance-regular graph Γ with vertex set X, diameter $D \geq 4$, and valency $k \geq 3$. For $0 \leq i \leq D$, let $\Gamma_i(x)$ denote the set of vertices in X that are distance i from vertex x. We assume there exist scalars $r, s, t \in \mathbb{R}$, not all zero, such that

$$r|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| + s|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + t = 0$$

for all $x, y, z \in X$ with path-length distances $\partial(x, y) = 2$, $\partial(x, z) = 3$, $\partial(y, z) = 3$. Fix $x \in X$, and let Γ_2^2 denote the graph with vertex set $\tilde{X} = \{y \in X \mid \partial(x, y) = 2\}$ and edge set $\tilde{R} = \{yz \mid y, z \in \tilde{X}, \partial(y, z) = 2\}$. We show that the adjacency matrix of the local graph Γ_2^2 has at most four distinct eigenvalues. We are motivated by the fact that our assumption above holds if Γ is Q-polynomial.

1. Introduction

Let $\Gamma = (X, R)$ denote a bipartite distance-regular graph with valency $k \geq 3$ and diameter $D \geq 4$. For $x \in X$ and $0 \leq i \leq D$ let $\Gamma_i(x)$ denote the set of vertices in X at distance i from x. For $x \in X$, let $\Gamma_2^2(x)$ denote the graph with vertex set $\tilde{X} = \{y \in X \mid \partial(x, y) = 2\}$ and edge set $\tilde{R} = \{yz \mid y, z \in \tilde{X}, \partial(y, z) = 2\}$. In this paper we prove the following theorem.

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Theorem 1.1. *Assume there exist scalars $r, s, t \in \mathbb{R}$, not all zero, such that for all vertices $x, y, z \in X$ with $\partial(x, y) = 2$, $\partial(x, z) = 3$, and $\partial(y, z) = 3$, we have*

$$r|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| + s|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| + t = 0. \tag{1}$$

Then for all $x \in X$, the adjacency matrix of the local graph $\Gamma_2^2(x)$ has at most four distinct eigenvalues.

Moreover, when Theorem 1.1 holds, we compute solutions for r, s, t in terms of the intersection numbers of Γ. Using results of Curtin [5], we give the eigenvalues of the local graph in terms of the intersection numbers of Γ. We conclude with a conjecture for further research.

We are motivated by the fact that when Γ is Q-polynomial, Miklavic has shown the assumptions of Theorem 1.1 hold [13, Theorem 9.1]. Furthermore, in [5, Section 5], Curtin assumes the local graphs $\Gamma_2^2(x)$ of a bipartite distance-regular graph have at most four distinct eigenvalues, and he outlines the resulting combinatorial implications. Here, we take a different approach: we assume Γ satisfies the combinatorial property from Theorem 1.1, and we will prove that the local graphs have at most four distinct eigenvalues. We note that the 2-homogeneous bipartite distance-regular graphs of Curtin [3] and Nomura [14] and the taut graphs of MacLean [8] satisfy the assumptions of Theorem 1.1.

We remark that this paper is part of a continuing effort to understand and classify the bipartite distance-regular graphs with at most two irreducible Terwilliger algebra modules of endpoint two, both of which are thin. In [4,11,12], Curtin, MacLean and Terwilliger show that the local graph eigenvalues determine the isomorphism class and the structure of these modules. Please see [6,8,9,11,10] for more work from this ongoing project.

2. Preliminaries

In this section, we review some basic definitions and results. For more information, the reader may consult the books of Bannai and Ito [1], Brouwer, Cohen, and Neumaier [2], and Godsil [7].

Throughout this paper, let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We refer to D as the *diameter* of Γ. For all $x \in X$ and for all integers i, we set $\Gamma_i(x) := \{y \in X \mid \partial(x, y) = i\}$.

The graph Γ is said to be *distance-regular* whenever for all integers h, i, j $(0 \le h, i, j \le D)$, and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of the choice of x and y. The numbers p^h_{ij} are called *intersection numbers* of Γ. It is conventional to abbreviate $c_i = p^i_{1i-1}$ $(1 \le i \le D)$, $a_i = p^i_{1i}$ $(0 \le i \le D)$, $b_i = p^i_{1i+1}$ $(0 \le i \le D-1)$, and to define $c_0 = 0$, $b_D = 0$. We note $c_1 = 1$ and abbreviate $\mu = c_2$.

For the rest of this paper we assume Γ is distance-regular with diameter D. We observe Γ is regular with valency $k = b_0$ and that $c_i + a_i + b_i = k$ $(0 \le i \le D)$. Moreover $b_i > 0$ $(0 \le i \le D-1)$ and $c_i > 0$ $(1 \le i \le D)$. For $0 \le i \le D$ we abbreviate $k_i = p^0_{ii}$. By [1, p. 195] we have

$$k_i = \frac{b_0 b_1 \ldots b_{i-1}}{c_1 c_2 \ldots c_i} \quad (1 \le i \le D). \tag{2}$$

We now consider the case in which Γ is bipartite. In the rest of this section, we recall some routine facts that will be useful later in the paper. To avoid trivialities, we will generally assume $D \ge 4$.

Lemma 2.1 ([2, Proposition 4.2.2]). *Let Γ denote a distance-regular graph with diameter $D \ge 4$ and valency k. The following are equivalent.*

(i) *Γ is bipartite.*
(ii) *$c_i + b_i = k$ $(0 \le i \le D)$.*