European Journal of Combinatorics 40 (2014) 42–54

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Direct and inverse problems in additive number theory and in non-abelian group theory

G.A. Freiman [a], M. Herzog [a], P. Longobardi [b], M. Maj [b], Y.V. Stanchescu [c,d]

[a] School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel

[b] Dipartimento di Matematica, Universita' di Salerno, 84084 Fisciano (Salerno), Italy

[c] Afeka Academic College, Tel Aviv 69107, Israel

[d] The Open University of Israel, Raanana 43107, Israel

ARTICLE INFO

Article history:
Received 17 September 2013
Accepted 6 February 2014
Available online 12 March 2014

ABSTRACT

We obtain new direct and inverse results for Minkowski sums of dilates and we apply them to solve certain direct and inverse problems in Baumslag–Solitar groups, assuming appropriate small doubling properties.

1. Introduction

The aim of this paper is threefold:

(a) Finding new direct and inverse results in the additive number theory concerning Minkowski sums of dilates.
(b) Finding a connection between the above results and some direct and inverse problems in the theory of Baumslag–Solitar (non-abelian) groups.
(c) Solving certain inverse problems in Baumslag–Solitar groups, assuming appropriate small doubling properties.

We start with our first topic (a), concerning the additive number theory. In this paper $\mathbb{Z}$ denotes the rational integers, $\mathbb{N}$ denotes the *non-negative* elements of $\mathbb{Z}$ and the size of a finite set A will be

E-mail addresses: grisha@post.tau.ac.il (G.A. Freiman), herzogm@post.tau.ac.il (M. Herzog), plongobardi@unisa.it (P. Longobardi), mmaj@unisa.it (M. Maj), yonis@afeka.ac.il, ionut@openu.ac.il (Y.V. Stanchescu).

http://dx.doi.org/10.1016/j.ejc.2014.02.001
0195-6698/

denoted by $|A|$. Subsets of $\mathbb{Z}$ of the form

$$r * A = \{rx : x \in A\},$$

where r is a positive integer and A is a *finite* subset of $\mathbb{Z}$, are called *r-dilates*.

Minkowski sums of dilates are defined as follows:

$$r_1 * A + \cdots + r_s * A = \{r_1x_1 + \cdots + r_sx_s : x_i \in A,\ 1 \le i \le s\}.$$

These sums have been recently studied in different situations by Bukh, Cilleruelo, Hamidoune, Ljujić, Nathanson, Plagne, Pontiveros, Rué, Serra, Silva and Vinuesa (see [2–4,9,10,12–15]). In particular, they examined sums of two dilates of the form

$$A + r * A = \{a + rb \mid a, b \in A\}$$

and solved various direct and inverse problems concerning their sizes.

For example, it was shown in [9,4] that

$$|A + 2 * A| \ge 3|A| - 2,$$

which represents a direct result. Moreover, they solved the following inverse problem: what is the *structure* of the set A if

$$|A + 2 * A| = 3|A| - 2?$$

Their answer was that in such case A must be an arithmetic progression.

Inverse problems of this type, where the exact bound is assumed, will be called *ordinary inverse problems*. The term *extended inverse problem* will refer to inverse problems in which a small diversion from the exact bound is allowed, still enabling us to reach a definite conclusion concerning the structure of A.

As an example of an extended inverse problem, consider the following question: what is the structure of the set A if $|A| \ge 3$ and

$$|A + 2 * A| < 4|A| - 4?$$

Our answer to this question is:

(A) *If $|A| \ge 3$ and $|A + 2 * A| < 4|A| - 4$, then A is a* subset *of an arithmetic progression P of size* $|P| \le |A + 2 * A| - 2|A| + 2 \le 2|A| - 3$ (see Theorem 4, Section 3).

The above mentioned authors and others studied also the sums $A + r * A$ for $r \ge 3$. In this direction we proved the following new (direct) result:

(B) *If $r \ge 3$, then* $|A + r * A| \ge 4|A| - 4$ (see Theorem 5, Section 4).

This very useful result yields a *uniform* bound for all sets A and for $r \ge 3$. In the literature, most bounds of this type are asymptotic.

It is worthwhile to notice that in Corollary 3.3 of [10] Hamidoune and Rué proved that $|n * A + m * A| \ge 4|A| - 4$. But they assume that $2 \le n < m$, with n and m coprime. As far as we can see, our result does not follow from their corollary.

We continue now with the second topic (b), dealing with a connection, noticed by us, between results concerning sums of dilates and some problems in the theory of Baumslag–Solitar groups.

If S and T are subsets of a group G, their *product* is defined as follows:

$$ST = \{st \mid s \in S,\ t \in T\}.$$

In particular, $S^2 = \{s_1s_2 \mid s_1, s_2 \in S\}$ and if $b \in G$, then $bS = \{bs \mid s \in S\}$.

For integers m and n, the general Baumslag–Solitar group $BS(m, n)$ is a group with two generators a, b and one defining relation $b^{-1}a^mb = a^n$:

$$BS(m, n) = \langle a, b \mid a^mb = ba^n\rangle.$$