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Clique versus independent set

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ABSTRACT

Yannakakis' Clique versus Independent Set problem (CL–IS) in communication complexity asks for the minimum number of cuts separating cliques from stable sets in a graph, called CS-separator. Yannakakis provides a quasi-polynomial CS-separator, i.e. of size $O(n^{\log n})$, and addresses the problem of finding a polynomial CS-separator. This question is still open even for perfect graphs. We show that a polynomial CS-separator almost surely exists for random graphs. Besides, if H is a split graph (i.e. has a vertex-partition into a clique and a stable set) then there exists a constant c_H for which we find a $O(n^{c_H})$ CS-separator on the class of H-free graphs. This generalizes a result of Yannakakis on comparability graphs. We also provide a $O(n^{c_k})$ CS-separator on the class of graphs without induced path of length k and its complement. Observe that on one side, c_H is of order $O(|H| \log |H|)$ resulting from Vapnik–Chervonenkis dimension, and on the other side, c_k is a tower function, due to an application of the regularity lemma.

One of the main reason why Yannakakis' CL–IS problem is fascinating is that it admits equivalent formulations. Our main result in this respect is to show that a polynomial CS-separator is equivalent to the polynomial Alon–Saks–Seymour Conjecture, asserting that if a graph has an edge-partition into k complete bipartite graphs, then its chromatic number is polynomially bounded in terms of k. We also show that the classical approach to the stubborn problem (arising in CSP) which consists in covering the set of all solutions by $O(n^{\log n})$ instances of 2-SAT is again equivalent to the existence of a polynomial CS-separator.

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1. Introduction

The goal of this paper is twofold. First, we focus on the Clique–Stable Set separation problem and provide classes of graphs for which polynomial separators exist. Then we show that this classical problem from communication complexity is equivalent to one in graph theory and one in CSP. Let us make a brief overview of each domain focusing on the problem.

Communication complexity and the Clique–Stable Set separation. A *clique* is a complete induced subgraph and a *stable set* is an induced subgraph with no edge. Yannakakis introduced in [27] the following communication complexity problem, called *Clique versus Independent Set* (CL–IS for brevity): given a publicly known graph Γ on n vertices, Alice and Bob agree on a protocol, then Alice is given a clique and Bob is given a stable set. They do not know which clique or which stable set was given to the other one, and their goal is to decide whether the clique and the stable set intersect or not, by minimizing the worst-case number of exchanged bits. Note that the intersection of a clique and a stable set is at most one vertex. In the deterministic version, Alice and Bob send alternatively messages one to each other, and the minimization is on the number of bits exchanged between them. It is a long standing open problem to prove a $\mathcal{O}(\log^2 n)$ lower bound for the deterministic communication complexity. In the non-deterministic version, a prover knowing the clique and the stable set sends a certificate in order to convince both Alice and Bob of the right answer. Then, Alice and Bob exchange one final bit, saying whether they agree or disagree with the certificate. The aim is to minimize the size of the certificate.

In this particular setting, a certificate proving that the clique and the stable set intersect is just the name of the vertex in the intersection. Such a certificate clearly has logarithmic size. Convincing Alice and Bob that the clique and the stable set do not intersect is much more complicated. A certificate can be a bipartition of the vertices such that the whole clique is included in the first part, and the whole stable set is included in the other part. Such a partition is called a cut that separates the clique and the stable set. A family $\mathcal{F}$ of m cuts such that for every disjoint clique and stable set, there is a cut in $\mathcal{F}$ that separates the clique and the stable set is called a CS-separator of size m. Observe that Alice and Bob can agree on a CS-separator at the beginning, and then the prover just gives the name of a cut that separates the clique and the stable set: the certificate has size $\log_2 m$. Hence if there is a CS-separator of polynomial size in n, one can ensure a non-deterministic certificate of size $\mathcal{O}(\log_2 n)$.

Yannakakis proved that there is a $c \log_2 n$ certificate for the CL–IS problem if and only if there is a CS-separator of size n^c. The existence of such a CS-separator is called in the following the Clique–Stable Set separation problem. The best upper bound so far, due to Hajnal (cited in [21]), is the existence for every graph G of a CS-separator of size $n^{(\log n)/2}$. The CL–IS problem arises from an optimization question which was studied both by Yannakakis [27] and by Lovász [22]. The question is to determine if the stable set polytope of a graph is the projection of a polytope in higher dimension, with a polynomial number or facets (called extended formulation). The existence of such a polytope in higher dimension implies the existence of a polynomial CS-separator for the graph. Moreover, Yannakakis proved that the answer is positive for several subclasses of perfect graphs, such as comparability graphs and their complements, chordal graphs and their complements, and Lovász proved it for a generalization of series–parallel graphs called t-perfect graphs. The existence of an extended formulation for general graphs has recently been disproved by Fiorini et al. [13], and is still open on perfect graphs.

Graph coloring and the Alon–Saks–Seymour conjecture. Given a graph G, the bipartite packing number, denoted by **bp**, is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of G. The Alon–Saks–Seymour conjecture (cited in [17]) states that if a graph has bipartite packing number k, then its chromatic number χ is at most $k + 1$. It is inspired from the Graham–Pollak theorem [14] which states that $\mathbf{bp}(K_n) = n - 1$. Huang and Sudakov proposed in [16] a counterexample to the Alon–Saks–Seymour conjecture (then generalized in [8]), twenty-five years after its statement. Actually they proved that there is an infinite family of graphs for which $\chi \geq \Omega(\mathbf{bp}^{6/5})$. The Alon–Saks–Seymour conjecture can now be restated as the *polynomial* Alon–Saks–Seymour conjecture: is the chromatic number polynomially upper bounded in terms of **bp**? Moreover, Alon and Haviv [3] observed that a gap $\chi \geq \Omega(\mathbf{bp}^c)$ for some graphs would imply a $\Omega(n^c)$ lower bound for the Clique–Stable Set separation problem. Consequently, Huang and Sudakov's