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On the location of roots of graph polynomials

Johann A. Makowsky [a], Elena V. Ravve [b], Nicolas K. Blanchard [c]

[a] *Computer Science Department, Technion—IIT, Haifa, Israel*

[b] *Software Engineering Department, ORT-Braude College, Karmiel, Israel*

[c] *École Normale Supérieure, Paris, France*

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ABSTRACT

Roots of graph polynomials such as the characteristic polynomial, the chromatic polynomial, the matching polynomial, and many others are widely studied. In this paper we examine to what extent the location of these roots reflects the graph theoretic properties of the underlying graph.

1. Introduction

A graph $G = (V(G), E(G))$ is given by the set of vertices $V(G)$ and a symmetric edge-relation $E(G)$. We denote by $n(G)$ the number of vertices, and by $m(G)$ the number of edges. $k(G)$ denotes the number of connected components of G. We denote the class of finite graphs by $\mathcal{G}$.

Graph polynomials are graph invariants with values in a polynomial ring, usually $\mathbb{Z}[\bar{X}]$ with $\bar{X} = (X_1, \dots, X_r)$. Let $P(G; \bar{X})$ be a graph polynomial. A graph G is P-unique if for all graphs G' the identity of $P(G; \bar{X})$ and $P(G'; \bar{X})$ implies that G is isomorphic to G'. As a graph invariant $P(G; \bar{X})$ can be used to check whether two graphs are not isomorphic. For P-unique graphs G and G' the polynomial $P(G; \bar{X})$ can also be used to check whether they are isomorphic. One usually compares graph polynomials by their *distinctive power*.

With our definition of graph polynomials there are too many graph polynomials. Traditionally, graph polynomials studied in the literature are definable in some logical formalisms. However, in this paper we only assume that our multivariate graph polynomials are of the form

$$P(G; \bar{X}) = \sum_{i_1,\dots,i_r=0}^{s(G)} h_{i_1,\dots,i_r}(G) X_1^{i_1} \cdot \dots \cdot X_r^{i_r}$$

where $\bar{X} = (X_1, \dots, X_r)$, $s(G)$ is a graph parameter with non-negative integers as its values, and $h_{i_1,\dots,i_r}(G) : i_1, \dots, i_r \leq s(G)$ are integer valued graph parameters. All graph polynomials in the

E-mail addresses: janos@cs.technion.ac.il (J.A. Makowsky), cselena@braude.ac.il (E.V. Ravve), koliaza@gmail.com (N.K. Blanchard).

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literature are of the above form.[1] The logical formalism is not needed for the results in this paper, and introducing it here would only make the paper less readable. Nevertheless, we shall indicate for the logically minded where the definability requirements can be added without changing the results.

1.1. Why study graph polynomials?

The first graph polynomial, the chromatic polynomial, was introduced in 1912 by G. Birkhoff to study the Four Color Conjecture. The emergence of the Tutte polynomial can be seen as an attempt to generalize the chromatic polynomial. The characteristic polynomial and the matching polynomial were introduced with applications from chemistry in mind. Physicists study various partition functions in statistical mechanics, in percolation theory and in the study of phase transitions. It turns out that many partition functions are incarnations of the Tutte polynomial. Another incarnation of the Tutte polynomial is the Jones polynomial in Knot Theory. The various incarnations of the Tutte polynomial triggered an interest in other graph polynomials. These graph polynomials are studied for various reasons:

- Graph polynomials can be used to distinguish non-isomorphic graphs.
  A graph polynomial is complete if it distinguishes all non-isomorphic graphs. The quest for a complete graph polynomial which is also easy to compute failed so far for two reasons. Either there were too many non-isomorphic graphs which could not be distinguished, and/or the proposed graph polynomial was more difficult to compute than just checking graph isomorphism.
- New graph polynomials are used to model behavior of physical, chemical or biological systems. The arguments whether a graph polynomial is interesting, depends on its success in predicting the behavior of the modeled systems.
- New graph polynomials are also studied as part of graph theory proper. Here one is interested in the interrelationship between various graph parameters without particular applications in mind. A graph polynomial is considered interesting, if many graph parameters can be (easily) derived from it.

This paper deals only with the graph theoretic aspect of graph polynomials, discarding the graph isomorphisms problem and modeling of systems describing phenomena in the natural sciences. It ultimately asks the question: When is a newly introduced graph polynomial interesting and deserves to be studied, and what aspects are more rewarding in this study than others. In particular we scrutinize the role of the location of the roots of specific graph polynomials in terms of other graph theoretic properties.

1.2. Equivalence of graph polynomials

Two graphs G_1 and G_2 are called *similar* if they have the same number of vertices, edges and connected components. Two graph polynomials $P(G; X_1, \dots, X_r)$ and $Q(G; Y_1, \dots, Y_s)$ are *equivalent in distinctive power (d.p.-equivalent)* if for every two similar graphs G_1 and G_2

$$P(G_1; X_1, \dots, X_r) = P(G_2; X_1, \dots, X_r) \quad \text{iff} \quad Q(G_1; Y_1, \dots, Y_s) = Q(G_2; Y_1, \dots, Y_s).$$

For a ring $\mathcal{R}$ let $\mathcal{R}^\infty$ denote the set of finite sequences of elements of $\mathcal{R}$. For a graph polynomial $P(G; \bar{X})$ we denote by $cP(\bar{X}) \in \mathbb{Z}^\infty$ the sequence of coefficients of $P(G; X)$. In Section 2 we will prove the following theorem and some variations thereof:

Proposition 1.1. *Two graph polynomials $P(G; X_1, \dots, X_r)$ and $Q(G; Y_1, \dots, Y_s)$ are d.p.-equivalent iff there are two functions $F_1, F_2 : \mathbb{Z}^\infty \to \mathbb{Z}^\infty$ such that for every graph G*

$$F_1(n(G), m(G), k(G), cP(G)) = cQ(G) \quad \textit{and}$$
$$F_2(n(G), m(G), k(G), cQ(G)) = cP(G).$$

[1] In [36,7,31,24] the class of graph polynomials definable in Second Order Logic **SOL** is studied, which imposes that $s(G)$ and $h_i(G) = h(G; i)$ are definable in **SOL**, which is stronger restriction.