European Journal of Combinatorics 41 (2014) 20–34

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Rectangular tileability and complementary tileability are undecidable

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ARTICLE INFO

Article history:
Received 13 December 2012
Accepted 11 March 2014
Available online 12 April 2014

ABSTRACT

Does a given set of polyominoes tile some rectangle? We show that this problem is undecidable. In a different direction, we also consider tiling a cofinite subset of the plane. The tileability is undecidable for many variants of this problem. However, we present an algorithm for testing whether the complement of a finite region is tileable by a set of rectangles.

1. Introduction

Tileability on the plane has been a subject of much study [18]. A lot of focus was in tilings on the square grid by polyominoes [14], including establishing NP-completeness [24,28,33] and finding efficient algorithms when possible [20,38]. Aperiodicity in tilings of the entire plane has also been well studied [29,35], with connections to ergodic theory [37,8] and quasicrystals [11]. Recently, a single (disconnected) tile that exhibits aperiodic behavior was found [40], partially settling a famous open problem.

Can the plane be tiled using translated copies of a given set of polyomino tiles? Berger showed that this decision problem is undecidable [5], meaning that there is no general algorithm that can always answer this question from the input. This implies that there exists *aperiodic tilesets*, *i.e.*, tiles that can *only* tile the plane without translational symmetry. Indeed, Berger provided an aperiodic tileset of 20 426 tiles, and Robinson reduces this number to 6 if rotations and reflections are allowed in addition to translations [39]. This disproves a conjecture of Wang (see Section 6.5).

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http://dx.doi.org/10.1016/j.ejc.2014.03.008
0195-6698/

To have aperiodicity and undecidability, clearly the complexity of tiles and tilings must increase without bound. The following result shows that one can encode this complexity in a single tile alone.[1]

Theorem 1.1 (*Complementary Tileability*). *There is a tileset* **T** *such that it is undecidable whether the complement of a finite input region* Γ *is tileable by* **T**.

Instead of tiling the entire plane or a cofinite subset, we also consider tiling finite regions. If a rectangle is tileable, then of course the plane is tileable. Thus the following is a variation of the result above.

Theorem 1.2 (*Rectangular Tileability*). *It is undecidable whether a given tileset* **T** *can tile* some *rectangle.*

This should be contrasted with tiling a given rectangle by a fixed tileset:

Theorem 1.3 ([23]). *Tileability of an* $[n \times m]$ *rectangle by a fixed tileset can be determined in time* $O(\log n + \log m)$.

We discuss this connection and some curious consequences of Theorem 1.2 in Section 6.3. It is worth noting that if we are given a finite region to tile as opposed to the entire plane, the problem is decidable simply with exhaustive search (see Section 6.2 for results in this finite setting). However, although the region to be tiled in Rectangular Tileability is finite, the problem is (potentially) undecidable as the finite region is unspecified. That is, we are tiling a finite object from an infinite collection. Indeed, Theorem 1.2 shows that Rectangular Tileability is undecidable. We prove this in Section 3, where we moreover show that the problem remains undecidable when the size $|\mathbf{T}|$ of the tileset is fixed. Also, as a corollary, we see that tileability of a unit square by finitely many similar copies of tiles (where rotations, reflections, and dilations are allowed in addition to translations) is also undecidable.

In Section 5, we prove undecidability for Augmentability, where we are to *augment* a finite simply connected region Γ by tiles so that the union is tileable. Any augmentable region Γ by the (horizontal and vertical) dominoes must necessarily be *balanced*, *i.e.*, has the same number of black and white squares when the plane is colored as a checkerboard. Korn showed that this is not sufficient, and also proved that Γ is augmentable if it is *row-convex*, *i.e.*, each horizontal row forms a single contiguous region [22, §11]. This should be compared with Theorem 1.4.

Usually once a result is established for a decision problem with several inputs, one may consider fixing some of these inputs and aim to obtain the same conclusion. To that end, we fix the tileset and instead let the region vary as the input of Tileability. However, decidability makes sense only if the input is finite, yet the region is infinite. As such, in Section 4, we consider some variations of tiling cofinite regions. Though most of these problems are undecidable, in the positive direction, we provide an algorithm for Tileability of cofinite regions by arbitrary sets of rectangular tiles.

Theorem 1.4. *It is decidable whether the complement of a given finite region* Γ *is tileable by a given tileset* **T** *consisting only of rectangles.*

In contrast, we mention in Section 6.1 that Tileability of *indented quadrants* by rectangles is undecidable.

2. Basic definitions

We call a subset of $\mathbb{Z}^2$ a *region*. By identifying $\mathbb{Z}^2$ as a union of closed unit squares in $\mathbb{R}^2$ centered at the integer lattice points, a region takes on a geometric *shape* in the obvious manner. We freely switch between viewpoints when convenient.[2] We say a finite region is an (*polyomino*) *tile* if its shape

[1] We require that this tile be used precisely once. To that end, we consider it as an input and tile its complement by a fixed tileset.

[2] For example, *finite* and *disjoint* refer to regions as subsets of $\mathbb{Z}^2$, so the shapes of disjoint regions (*e.g.*, tiles in a tiling) may intersect on their boundaries, but *simply connected* refers to the shapes of regions.