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Unavoidable vertex-minors in large prime graphs

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ABSTRACT

A graph is *prime* (with respect to the split decomposition) if its vertex set does not admit a partition (A, B) (called a *split*) with $|A|$, $|B| \geqslant 2$ such that the set of edges joining A and B induces a complete bipartite graph.

We prove that for each n, there exists N such that every prime graph on at least N vertices contains a vertex-minor isomorphic to either a cycle of length n or a graph consisting of two disjoint cliques of size n joined by a matching.

1. Introduction

In this paper, all graphs are simple and undirected. We write P_n and C_n to denote a graph that is a path and a cycle on n vertices, respectively. We aim to find analogues of the following theorems.

- (Ramsey's theorem)
  For every n, there exists N such that every graph on at least N vertices contains an induced subgraph isomorphic to K_n or $\overline{K_n}$.
- (folklore; see Diestel's book [8, Proposition 9.4.1])
  For every n, there exists N such that every *connected* graph on at least N vertices contains an induced subgraph isomorphic to K_n, $K_{1,n}$, or P_n.
- (folklore; see Diestel's book [8, Proposition 9.4.2])
  For every n, there exists N such that every 2-*connected* graph on at least N vertices contains a topological minor isomorphic to C_n or $K_{2,n}$.
- (Oporowski, Oxley, and Thomas [15])

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For every n, there exists N such that every 3-*connected* graph on at least N vertices contains a minor isomorphic to the wheel graph W_n on n vertices or $K_{3,n}$.

- (Ding, Chen [9])
For every integer n, there exists N such that every *connected and co-connected* graph on at least N vertices contains an induced subgraph isomorphic to P_n, $K^s_{1,n}$ (the graph obtained from $K_{1,n}$ by subdividing one edge once), $K_{2,n}\backslash e$, or $K_{2,n}/e\backslash f\backslash g$ where $\{f, g\}$ is a matching in $K_{2,n}/e$. A graph is *co-connected* if its complement graph is connected.
- (Chun, Ding, Oporowski, and Vertigan [6])
For every integer $n \geqslant 5$, there exists N such that every internally 4-connected graph on at least N vertices contains a parallel minor isomorphic to K_n, $K'_{4,n}$ ($K_{4,n}$ with a complete graph on the vertices of degree n), TF_n (the n-partition triple fan with a complete graph on the vertices of degree n), D_n (the n-spoke double wheel), D'_n (the n-spoke double wheel with axle), M_n (the $(2n+1)$-rung Mobius zigzag ladder), or Z_n (the $(2n)$-rung zigzag ladder).

These theorems commonly state that every sufficiently large graph having certain connectivity contains at least one graph in the list of *unavoidable* graphs by certain graph containment relation. Moreover in each theorem, the list of unavoidable graphs is *optimal* in the sense that each unavoidable graph in the list has the required connectivity, can be made arbitrary large, and does not contain other unavoidable graphs in the list.

In this paper, we discuss *prime* graphs as a connectivity requirement. A *split* of a graph G is a partition (A, B) of the vertex set $V(G)$ having subsets $A_0 \subseteq A$, $B_0 \subseteq B$ such that $|A|, |B| \geqslant 2$ and a vertex $a \in A$ is adjacent to a vertex $b \in B$ if and only if $a \in A_0$ and $b \in B_0$. This concept was first studied by Cunningham [7] in his research on split decompositions. We say that a graph is *prime* if it has no splits. Sometimes we say a graph is *prime with respect to split decomposition* to distinguish with another notion of primeness with respect to modular decomposition.

Prime graphs play an important role in the study of circle graphs (intersection graphs of chords in a circle) and their recognition algorithms. Bouchet [2], Naji [14], and Gabor, Hsu, and Supowit [11] independently showed that prime circle graphs have a unique chord diagram. This is comparable to the fact that 3-connected planar graphs have a unique planar embedding.

The graph containment relation which we will mainly discuss is called a *vertex-minor*. A graph H is a *vertex-minor* of a graph G if there exist a sequence $v_1, v_2, \ldots, v_n$ of (not necessarily distinct) vertices and a subset $X \subseteq V(G)$ such that $H = G * v_1 * v_2 \cdots * v_n \backslash X$, where $G * v$ is an operation called *local complementation*, to take the complement graph only in the neighborhood of v. The detailed description will be given in Section 2.1. Vertex-minors are important in circle graphs; for instance, Bouchet [5] proved that a graph is a circle graph if and only if it has no vertex-minor isomorphic to one of three particular graphs.

Prime graphs have been studied with respect to vertex-minors, perhaps because local complementation preserves prime graphs, shown by Bouchet [2]. In addition, he showed the following.

Theorem 1.1 (*Bouchet [2]*). *Every prime graph on at least* 5 *vertices must contain a vertex-minor isomorphic to* C_5.

Here is the main theorem of this paper.

Theorem 7.1. *For every n, there is N such that every prime graph on at least N vertices has a vertex-minor isomorphic to* C_n *or* $K_n \boxminus K_n$.

The graph $K_n \boxminus K_n$ is a graph obtained by joining two copies of K_n by a matching of size n, see Fig. 1. This notation will be explained in Section 2.4. In addition, we show that this list of unavoidable vertex-minors in Theorem 7.1 is optimal, which will be discussed in Section 8. We will heavily use Ramsey's theorem iteratively and so our bound N is astronomical in terms of n.

The proof consists of two parts.

(1) We first prove that for each n, there exists N such that every prime graph having an induced path of length N contains a vertex-minor isomorphic to C_n. (In fact, we prove that $N = \lceil 6.75n^7 \rceil$.)
(2) Secondly, we prove that for each n, there exists N such that every prime graph on at least N vertices contains a vertex-minor isomorphic to P_n or $K_n \boxminus K_n$.