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# Symplectic graphs over finite commutative rings

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ABSTRACT

Let $V$ be a symplectic space over a finite commutative ring $R$ and let $\mathcal{G}_{\mathrm{Sp}_R(V)}$ be the symplectic graph over $R$. In this work, we show that it is arc transitive and determine the chromatic number. Moreover, if $R$ is a finite local ring, we obtain its automorphism group, and the chromatic number and the automorphism group of each subconstituent.



## 1. Introduction

The general symplectic graph associated with nonsingular alternate matrices over a field is studied by Tang and Wan [11] as a new family of strongly regular graphs. Their work used orthogonal complements and matrix theory on symmetric bilinear forms over a finite field. Meemark and Prinyasart [9] introduced the symplectic graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ for a symplectic space $V$ over a commutative ring $R$. They showed that their symplectic graph is arc transitive when $R = \mathbb{Z}_{p^n}$, $p$ is an odd prime and $n \geq 1$. There are some articles influenced by this definition such as [7,8,4,3]. Mostly, the work was on strong regularity, automorphism groups, vertex and arc transitivities, chromatic numbers and subconstituents of symplectic graphs over a finite field, modulo $p^n$, and modulo $pq$, where $p$ and $q$ are primes and $n \geq 1$. Recently, the authors [10] studied those topics for symplectic graphs over a finite local ring and obtained results parallel to [11,9,7,8,3]. They applied a combinatorial approach similar to [9] and some basic properties of a local ring.

In what follows, we work on the symplectic graph over a finite commutative ring. We show that it is arc transitive and determine the chromatic number. Moreover, if $R$ is a finite local ring, we

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obtain its automorphism group, and the chromatic number and the automorphism group of each subconstituent. We shall repeatedly refer the results in three major articles, namely, [10,11,6].

The paper is organized as follows. Basic results of symplectic graphs over a finite commutative ring are presented in Section 2. Next, in Section 3, we give results on the chromatic number for symplectic graphs over a finite local ring. Automorphisms and arc transitivities are discussed in Section 4. This section also includes the fractional chromatic number and the chromatic number of symplectic graphs over a finite commutative ring (Theorem 4.9). We study the subconstituents of symplectic graphs in the final section.

## 2. Symplectic graphs

Let $R$ be a commutative ring and let $V$ be a free $R$-module of rank $2\nu$, where $\nu \geq 1$. Assume that we have a function $\beta$: $V \times V \to R$ which is $R$-bilinear, $\beta(\vec{x}, \vec{x}) = 0$ for all $\vec{x}$ in $V$ and the $R$-module morphism from $V$ to $V^* = \mathrm{Hom}_R(V, R)$ given by $\vec{x} \mapsto \beta(\cdot, \vec{x})$ is an isomorphism. We call the pair $(V, \beta)$ a *symplectic space*. A vector $\vec{x}$ in $V$ is said to be *unimodular* if there is an $f$ in $V^*$ with $f(\vec{x}) = 1$; equivalently, if $\vec{x} = \alpha_1\vec{b}_1 + \cdots + \alpha_{2\nu}\vec{b}_{2\nu}$, where $\{\vec{b}_1, \ldots, \vec{b}_{2\nu}\}$ is a basis for $V$, then the ideal $(\alpha_1, \ldots, \alpha_{2\nu}) = R$. If $\vec{x}$ is unimodular, then the *line* $Rx$ is a free $R$-direct summand of rank one.

A *hyperbolic pair* $\{\vec{x}, \vec{y}\}$ is a pair of unimodular vectors in $V$ with the property that $\beta\left(\vec{x}, \vec{y}\right) = 1$. The module $H = R\vec{x} \oplus R\vec{y}$ is called a *hyperbolic plane*. Let $(V, \beta)$ be a symplectic space. An $R$-module automorphism $\sigma$ on $V$ is an *isometry* on $V$ if $\beta(\sigma(\vec{x}), \sigma(\vec{y})) = \beta\left(\vec{x}, \vec{y}\right)$ for all $\vec{x}, \vec{y} \in V$. The group of isometries on $V$ is called the *symplectic group of* $(V, \beta)$ *over* $R$ and denoted by $\mathrm{Sp}_R(V)$. Define the graph $\mathcal{G}_{\mathrm{Sp}_R(V)}$ with vertex set the set of lines $\{R\vec{x} : \vec{x}$ is a unimodular vector in $V\}$ and with adjacency given by

$$R\vec{x} \text{ is adjacent to } R\vec{y} \quad \text{if and only if} \quad \beta(\vec{x}, \vec{y}) \in R^{\times}.$$

Here, $R^{\times}$ denotes the group of invertible elements in $R$. We call $\mathcal{G}_{\mathrm{Sp}_R(V)}$, the *symplectic graph of* $(V, \beta)$ *over* $R$.

A *local ring* is a commutative ring which has a unique maximal ideal. Note that for a local ring $R$, its unique maximal ideal is given by $M = R \setminus R^{\times}$ and we call the field $R/M$, the *residue field of* $R$.

Let $R$ be a finite commutative ring. It is well known that any finite commutative ring is a product of finite local rings (Theorem 8.7 of [1]) and we have completely studied our graphs over a finite local ring (see Section 2 of [10]). Write

$$R = R_1 \times R_2 \times \cdots \times R_t$$

as a direct product of finite local rings $R_i$, $i = 1, 2, \ldots, t$. Consider $V = R^{2\nu}$, a free $R$-module of rank $2\nu$, where $\nu \geq 1$. We have the canonical 1–1 correspondence

$$\vec{x} = (x_1, x_2, \ldots, x_{2\nu}) \overset{\varphi}{\mapsto} ((x_1^{(j)})_{j=1}^t, (x_2^{(j)})_{j=1}^t, \ldots, (x_{2\nu}^{(j)})_{j=1}^t).$$

Note that if $\vec{x}, \vec{y} \in V$, then this correspondence induces the symplectic map $\beta$ on $V$ by

$$\begin{aligned}\beta(\vec{x}, \vec{y}) &= \beta(((x_1^{(j)})_{j=1}^t, (x_2^{(j)})_{j=1}^t, \ldots, (x_{2\nu}^{(j)})_{j=1}^t), ((y_1^{(j)})_{j=1}^t, (y_2^{(j)})_{j=1}^t, \ldots, (y_{2\nu}^{(j)})_{j=1}^t)) \\ &= (\beta_1(\vec{x}^{(1)}, \vec{y}^{(1)}), \beta_2(\vec{x}^{(2)}, \vec{y}^{(2)}), \ldots, \beta_t(\vec{x}^{(t)}, \vec{y}^{(t)})) \\ &= \Big(\sum_{i=1}^{\nu}(x_i^{(1)}y_{\nu+i}^{(1)} - x_{\nu+i}^{(1)}y_i^{(1)}), \sum_{i=1}^{\nu}(x_i^{(2)}y_{\nu+i}^{(2)} - x_{\nu+i}^{(2)}y_i^{(2)}), \ldots, \sum_{i=1}^{\nu}(x_i^{(t)}y_{\nu+i}^{(t)} - x_{\nu+i}^{(t)}y_i^{(t)})\Big),\end{aligned}$$

where $\vec{x}^{(j)} = (x_1^{(j)}, \vec{x}_2^{(j)}, \ldots, \vec{x}_{2\nu}^{(j)}) \in V^{(j)} := R_j^{2\nu}$ and $(V^{(j)}, \beta_j)$ is a symplectic space of $R_j$ of rank $2\nu$, for all $j = 1, 2, \ldots, t$. Since $R^{\times} = R_1^{\times} \times R_2^{\times} \times \cdots \times R_t^{\times}$, we have

$$\beta(\vec{x}, \vec{y}) \in R^{\times} \Leftrightarrow \sum_{i=1}^{\nu}(x_i^{(j)}y_{\nu+i}^{(j)} - x_{\nu+i}^{(j)}y_i^{(j)}) \in R_j^{\times} \quad \text{for all } j \in \{1, 2, \ldots, t\}. \tag{2.1}$$