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On Sperner (semi)hypergroups

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ABSTRACT

In this paper using Sperner family we introduce and study a new class of (semi)hypergroups that we call the class of Sperner (semi)hypergroups. We characterize strongly Sperner semihypergroups of order 3 and the Sperner Rosenberg hypergroups of order 3.

1. Introduction

The concept of a hypergroup which is based on the notion of hyperoperation was introduced by Marty in [7] and studied extensively by many mathematicians. Hypergroup theory extends some well-known group results and introduce new topics leading thus to a wide variety of applications, as well as to a broadening of the fields of investigation. Surveys of the theory can be found in the books of Corsini [1], Davvaz and Leoreanu-Fotea [4], Corsini and Leoreanu [2] and Vougiouklis [10]. In this paper using Sperner family we introduce and study some classes of (semi)hypergroups that we call them the class of row Sperner (semi)hypergroups of rank k, column Sperner (semi)hypergroups of rank m, Sperner (semi)hypergroups of rank t and strongly Sperner (semi)hypergroups, respectively. We characterize strongly Sperner semihypergroups of order 3 and the Sperner Rosenberg hypergroups of order 3 up to isomorphism. In the following we recall some basic notions of hyperstructures theory from [2].

Definition 1.1. A hypergroupoid is a nonempty set H together with a map $\cdot : \ H \times H \longrightarrow P^*(H)$ which is called hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H.

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Remark 1.2. Suppose that H is a nonempty set. A hyperoperation $\cdot : H \times H \longrightarrow P^*(H)$, yields an operation $\otimes : P^*(H) \times P^*(H) \longrightarrow P^*(H)$, defined by $A \otimes B = \bigcup_{a \in A, b \in B} a \cdot b$. Conversely an operation on $P^*(H)$ yields a hyperoperation on H, defined by $x \cdot y = \{x\} \otimes \{y\}$.

Definition 1.3. (i) A semihypergroup is a hypergroupoid $(H, \cdot)$ such that for all a, b and c in H we have $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(ii) A quasihypergroup is a hypergroupoid $(H, \cdot)$ which satisfies the reproductive law, i.e., for all a and b in H, $H \cdot a = a \cdot H = H$.
(iii) A hypergroup is a semihypergroup which is also a quasihypergroup.

Definition 1.4. A semihypergroup $(H, \circ)$ is called simplifiable on the left if the following implication valid:

$$\forall (x, a, b) \in\in H^3, \quad x \circ a \cap x \circ b \neq \varnothing \Longrightarrow a = b.$$

Similarly, we can define the simplifiability on the right. The semihypergroup $(H, \circ)$ is called simplifiable if it is simplifiable on the left and on the right.

F. Marty [7] proved that any hypergroup simplifiable on the left (or on the right) is a group. Later, M. Koskas [5] gave a simpler proof for the same result. In [6] it is proved the following:

Theorem 1.5. *Let $(H, \circ)$ be a semihypergroup such that $\forall t \in H, t \circ H = H$ and $\exists t_0 \in H, H \circ t_0 = H$.*
(i) *If H is simplifiable on the left, then H is a group;*
(ii) *If H is simplifiable on the right, then H is a group.*

Definition 1.6. A semihypergroup $(H, \cdot)$ is called *complete* if for all $n, m \geqslant 2$ and for all $(x_1, x_2, \ldots, x_n) \in H^n$ and $(y_1, y_2, \ldots, y_m) \in H^m$ we have the following implication:

$$\prod_{i=1}^{n} x_i \bigcap \prod_{j=1}^{m} y_j \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j,$$

where $\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n$.

Theorem 1.7 (*See [2]*)**.** *A (semi)hypergroup $(H, \circ)$ is complete if and only if $H = \cup A_s (s \in S)$, where S, A_s the following conditions are satisfied:*
(i) *$(S, \cdot)$ is (semi)group;*
(ii) *for all $(s, t) \in S^2$ such that $s \neq t$ have $A_s \cap A_t = \oslash$;*
(iii) *if $(a, b) \in A_s \times A_t$ then $a \circ b = A_{s \cdot t}$.*

2. Sperner hypergroups

Let $\mathcal{F}$ be a family of subsets of H. Then it is called a Sperner family if the following implication valid:

$$\forall (X, Y) \in \mathcal{F}^2, \quad [X \subseteq Y \text{ or } Y \subseteq X] \Rightarrow X = Y.$$

In this section we use the notion of Sperner family and we introduce and study some classes of (semi)hypergroups that we call them the class of row Sperner (semi)hypergroups of rank k, column Sperner (semi)hypergroups of rank m, Sperner (semi)hypergroups of rank t and strongly Sperner (semi)-hypergroups, respectively. The class of Sperner semihypergroups consists of the class of simplifiable semihypergroups which is introduced by F. Marty [7]. At the end we characterize strongly Sperner semihypergroups of order 3.

Let $H = \{x_1, x_2, \ldots, x_n\}$ and $(H, \circ)$ be a finite (semi)hypergroup. For every $(x_i, x_j) \in H^2$ we define $R_i = \{x_i \circ x_j | x_j \in H\}$ and $C_j = \{x_k \circ x_j | x_k \in H\}$. Now let r_i and c_j be cardinal numbers of biggest Sperner subsets of R_i and C_j, respectively.

Definition 2.1. A finite (semi)hypergroup H of order n (i.e., $|H| = n$) is called
(i) row Sperner of rank m if $Max\{r_i | 1 \leq i \leq n\} = m$;
(ii) column Sperner of rank k if $Max\{c_j | 1 \leq j \leq n\} = k$;