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Sign patterns of rational matrices with large rank

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ABSTRACT

Let A be a real matrix. The term rank of A is the smallest number t of lines (that is, rows or columns) needed to cover all the nonzero entries of A. We prove a conjecture of Li et al. stating that, if the rank of A exceeds $t-3$, there is a rational matrix with the same sign pattern and rank as those of A. We point out a connection of the problem discussed with the Kapranov rank function of tropical matrices, and we show that the statement fails to hold in general if the rank of A does not exceed $t-3$.

1. Introduction

The problem of constructing a matrix over a given ordered field with specified sign pattern and rank deserved a significant amount of attention in recent publications, see [2] and references therein. The present paper establishes a connection of this problem with that of computing certain rank functions arisen from tropical geometry. We prove the conjecture on sign patterns of rational matrices formulated in [2], and we present the examples showing the optimality of our result.

2. Preliminaries

The following notation is used throughout our paper. By $U^{m\times n}$ we denote the set of all m-by-n matrices with entries from a set U, by $A_{ij} \in U$ we denote an entry of a matrix $A \in U^{m\times n}$. By $U_{(i)}$ we denote the ith row of U, and we call a *line* of a matrix any of its columns or rows.

A field R is called *ordered* if, for some subset $P \subset R$ closed under addition and multiplication, the sets P, $-P$, and $\{0\}$ form a partition of R. The elements of P are then called *positive*, and those from

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$-P$ *negative*. The *sign pattern* of a matrix $A \in R^{m\times n}$ is the matrix $S = \mathcal{S}(A) \in \{+, -, 0\}^{m\times n}$ defined as $S_{ij} = +$ if A_{ij} is positive, $S_{ij} = -$ if A_{ij} is negative, and $S_{ij} = 0$ if $A_{ij} = 0$. The *minimum rank* of a sign pattern S with respect to R is the minimum of the ranks of matrices B over R satisfying $\mathcal{S}(B) = S$.

There are a significant number of recent publications devoted to the study of the minimal ranks of sign patterns (see [2] and references therein), and our paper aims to prove a conjecture formulated in [2]. This conjecture relates the minimal rank of a pattern to a concept of the *term rank* of a matrix, which is defined as the smallest number of lines needed to include all the nonzero elements of that matrix. The classical *König's theorem* states the term rank of a matrix A equals the maximum number of nonzero entries of A no two of which belong to the same line, so the term rank of a sign pattern S can be thought of as the maximum of the ranks of matrices C over R satisfying $\mathcal{S}(C) = S$. Now we can formulate the conjecture by Li et al. relating the concepts of minimum and term ranks for sign pattern matrices.

Conjecture 2.1 (*[2, Conjecture 4.2]*)**.** *Assume that S is a sign pattern matrix with term rank equal to t, and let r be the minimum rank of S over the reals. If $r \geqslant t-2$, then the minimum rank of S over the rationals is r as well.*

In Section 3 we develop a combinatorial technique which allows to prove Conjecture 2.1. In Section 4 we establish the connection of the problem discussed with the Kapranov rank function of Boolean matrices introduced in [1]. We also make the use of matroid theory to prove the optimality of the bound in Conjecture 2.1 by showing that its statement fails to hold in general if r is less than $t-2$.

3. Proof of the result

We start with two easy observations helpful for further considerations.

Observation 3.1. *Multiplying a row of a real matrix A by a nonzero number will not change the minimal ranks of its sign pattern.*

Proof. Trivial. □

Observation 3.2. *Let r and t be, respectively, the minimum and term ranks of a sign pattern S with respect to an ordered field R. Then, for any integer $h \in [r, t]$, there is a matrix over R which has rank h and sign pattern S.*

Proof. Changing a single entry produces a matrix whose rank differs by at most 1 from that of the initial matrix. □

The following lemma gives a useful description of the rank of a block matrix. We say that a linear subspace $S \subset \mathbb{R}^d$ is *rational* if S has a basis consisting of vectors that have rational coordinates only.

Lemma 3.3. *Let $V_1 \in \mathbb{Q}^{p\times(p-1)}$ and $V_2 \in \mathbb{Q}^{(q-1)\times q}$ be rational matrices that have ranks $p-1$ and $q-1$, respectively. Then the set $\mathcal{W}$ of all $W \in \mathbb{R}^{p\times q}$ for which the matrix $U = \begin{pmatrix} W & V_1 \\ V_2 & O \end{pmatrix}$ has rank $p+q-2$ is a rational subspace.*

Proof. Note that rational elementary transformations on the first p rows or first q columns of U cannot break the property of $\mathcal{W}$ to be a rational subspace. So we can assume that V_1 and V_2 differ from the identity matrices, respectively, by adding the pth zero row and the qth zero column. In this case, $\mathcal{W}$ consists of those matrices W which satisfy $W_{pq} = 0$. □

We need another lemma to prove Conjecture 2.1. By $[x]$ we denote the integer part of a real x.

Lemma 3.4. *Assume that a vector $a = (a_1, \ldots, a_n)$ and a matrix $B \in \mathbb{R}^{n\times 2}$ satisfy $aB = (0\,0)$. Assume that every entry of the first column of B is either 0 or 1. Define, for integer $N > 0$, the n-by-2 matrix $C = C(N)$ by $C_{ij} = [B_{ij}N]$. Then, for any sufficiently large N, there is a rational vector $x = x(N)$ satisfying $xC = (0\,0)$ and $x(N) \to a$ as $N \to \infty$.*