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On an extension of Riordan array and its application in the construction of convolution-type and Abel-type identities

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ABSTRACT

Using the basic fact that any formal power series over the real or complex number field can always be expressed in terms of given polynomials $\{p_n(t)\}$, where $p_n(t)$ is of degree n, we extend the ordinary Riordan array (resp. Riordan group) to a generalized Riordan array (resp. generalized Riordan group) associated with $\{p_n(t)\}$. As new application of the latter, a rather general Vandermonde-type convolution formula and certain of its particular forms are presented. The construction of the Abel type identities using the generalized Riordan arrays is also discussed.

1. Introduction

In the recent literature, special emphasis has been given to the concept of *Riordan arrays* associated with power series, which are a generalization of the well-known Pascal triangle. Riordan arrays are infinite, lower triangular matrices defined by the generating function (GF) of their columns. They form a group, called *the Riordan group* (cf. Shapiro, Getu, Woan, and Woodson [32]). Some of the main results on the Riordan group and its application to combinatorial sums and identities can be found in Sprugnoli [33,34], on subgroups of the Riordan group in Peart and Woan [24] and Shapiro [29], on some characterizations of Riordan matrices in Rogers [25], Merlini, Rogers, Sprugnoli, and Verri [21],

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and He and Sprugnoli [17], and on many interesting related results in Cheon, Kim, and Shapiro [2,3], Deutsch, Ferrari, and Rinaldi [6], Gould and He [8], He [9], He, Hsu, and Shiue [12], Hsu [18], Nkwanta [23], Shapiro [30,31], Wang and Wang [35], and so forth.

More formally, let us consider the set of formal power series (f.p.s.) $\mathcal{F} = \mathbb{R}[[t]]$; the *order* of $f(t) \in \mathcal{F}, f(t) = \sum_{k=0}^{\infty} f_k t^k$ ($f_k \in \mathbb{R}$), is the minimal number $r \in \mathbb{N}$ such that $f_r \neq 0$; $\mathcal{F}_r$ is the set of formal power series of order r. It is known that $\mathcal{F}_0$ is the set of *invertible* f.p.s. and $\mathcal{F}_1$ is the set of *compositionally invertible* f.p.s., that is, the f.p.s. $f(t)$ for which the compositional inverse $f^*(t)$ exists such that $f(f^*(t)) = f^*(f(t)) = t$. Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$; the pair $(d(t),\ h(t))$ defines the *(proper) Riordan array* $D = (d_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ having

$$d_{n,k} = [t^n]d(t)h(t)^k \tag{1}$$

or, in other words, having $d(t)h(t)^k$ as the GF whose coefficients make-up the entries of column k.

It is immediate to show that the usual row-by-column product of two Riordan arrays is also a Riordan array:

$$(d_1(t),\ h_1(t)) \cdot (d_2(t),\ h_2(t)) = (d_1(t)d_2(h_1(t)),\ h_2(h_1(t))). \tag{2}$$

The Riordan array $I = (1,\ t)$ is everywhere 0 except that it contains all 1's on the main diagonal; it is easily seen that I acts as an identity for this product, that is, $(1,\ t)\cdot(d(t),\ h(t)) = (d(t),\ h(t))\cdot(1, t) = (d(t),\ h(t))$. From these facts, we deduce a formula for the inverse Riordan array:

$$(d(t),\ h(t))^{-1} = \left(\frac{1}{d(h^*(t))},\ h^*(t)\right) \tag{3}$$

where $h^*(t)$ is the compositional inverse of $h(t)$. In this way, the set $\mathcal{R}$ of proper Riordan arrays is a group.

Let $d(t) \in \mathcal{F}_0$ and $h(t) \in \mathcal{F}_1$. Then the polynomials $u_n(x)$ $(n = 0, 1, 2, \ldots)$ defined by the GF

$$d(t)e^{xh(t)} = \sum_{n\geq 0} u_n(x)t^n \tag{4}$$

are called Sheffer-type polynomials with $u_0(x) = 1$. The set of all Sheffer-type polynomial sequences $\{u_n(x) = [t^n]d(t)e^{xh(t)}\}$ with an operation, "umbral composition" (*cf.* [26] and [28]), forms a group called the Sheffer group. [12] presents the isomorphism between the Riordan group and Sheffer group.

Rogers [25] introduced the concept of the *A-sequence* for Riordan arrays; Merlini, Rogers, Sprugnoli, and Verri [21] introduced the related concept of the *Z-sequence* and showed that these two concepts, together with the element $d_{0,0}$, completely characterize a proper Riordan array. He and Sprugnoli [17] presented the characterization of Riordan arrays by means of the A- and Z-sequences for some subgroups of $\mathcal{R}$ and the products and the inverses of Riordan arrays.

In [10], one of the authors defined the generalized Sheffer-type polynomial sequences as follows.

Definition 1.1 (*[10]*)**.** Let $d(t), U(t)$, and $h(t)$ be any formal power series over the real number field $\mathbb{R}$ or complex number field $\mathbb{C}$ with $d(0) = 1, U(0) = 1, h(0) = 0$, and $h'(0) \neq 0$. Then the polynomials $u_n(x)$ $(n = 0, 1, 2, \ldots)$ defined by the GF

$$d(t)U(xh(t)) = \sum_{n\geq 0} u_n(x)t^n \tag{5}$$

are called the generalized Sheffer-type polynomials associated with $(d(t), h(t))_{U(t)}$. Accordingly, $u_n(D)$ with $D \equiv d/dt$ is called Sheffer-type differential operator of degree n associated with $(d(t), h(t))_{U(t)}$. Particularly, $u_0(D) \equiv I$ is the identity operator due to $u_0(x) = 1$.

One of the authors [9] shows that for every $U(t)$ there exists a one-to-one correspondence between $(d(t), h(t))$ and $\{u_n(x)\}$, and the collection, P_U, of all polynomial sequences $\{u_n(x)\}$ with respect to $V(t) = \sum_{n\geq 0} a_n t^n$, defined by (5), forms a group $(P_U, \tilde{\#})$ under the operation $\tilde{\#}$, defined by

$$\{p_n(x)\}\tilde{\#}\{q_n(x)\} = \{r_n(x) = \sum_{k=0}^{n} r_{n,k}x^k : r_{n,k} = \sum_{\ell=k}^{n} p_{n,\ell}q_{\ell,k}/a_\ell,\ n \geq k\},$$