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Reflection arrangements and ribbon representations

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ABSTRACT

Ehrenborg and Jung (2011) recently related the order complex for the lattice of d-divisible partitions with the simplicial complex of *pointed ordered set partitions* via a homotopy equivalence. The latter has top homology naturally identified as a Specht module. Their work unifies that of Calderbank, Hanlon, Robinson (1986), and Wachs (1996). By focusing on the underlying geometry, we strengthen and extend these results from type A to all real reflection groups and the complex reflection groups known as *Shephard groups*.

1. Introduction

The aim of this paper is to elucidate a phenomenon that has been studied for the symmetric group $\mathfrak{S}_n$ by studying the underlying geometry. Here we sketch the phenomenon, along with our geometric interpretation and generalization.

For $n+1$ divisible by d, recall that the d-divisible partition lattice $\Pi^d_{n+1} \cup \{\hat{0}\}$ is the poset of partitions of the set $\{1, 2, \ldots, n+1\}$ with parts divisible by d, together with a unique minimal element $\hat{0}$ when $d > 1$. In [13], Calderbank, Hanlon and Robinson showed that for $d > 1$ the top homology of the order complex $\Delta(\Pi^d_{n+1} \setminus \{\hat{1}\})$, when restricted from $\mathfrak{S}_{n+1}$ to $\mathfrak{S}_n$, carries the *ribbon representation* of $\mathfrak{S}_n$ corresponding to a ribbon with row sizes $(d, d, \ldots, d, d-1)$. Wachs [33] gave a more explicit proof of this fact. Their results generalized Stanley's [28] result for the Möbius function of $\Pi^d_n \cup \{\hat{0}\}$, which in turn generalized G. S. Sylvester's [31] result for 2-divisible partitions $\Pi^2_n \cup \{\hat{0}\}$.

Ehrenborg and Jung extend the above results by introducing posets of *pointed partitions* $\Pi^\bullet_{\vec{c}}$ parametrized by a composition $\vec{c}$ of n with last part possibly 0, from which they obtain all ribbon representations. More importantly, they explain why Specht modules are appearing by establishing a homotopy equivalence with another complex whose top homology is naturally a Specht module.

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Ehrenborg and Jung construct their pointed partitions $\Pi^\bullet_{\vec{c}} \subset \Pi^\bullet_n \cong \Pi_{n+1}$ by distinguishing a particular block (called the *pointed block*) and restricting to those of type $\vec{c}$. They show that $\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\})$ is homotopy equivalent to a wedge of spheres, and that the top reduced homology $\widetilde{H}_{\mathrm{top}}(\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\}))$ is the $\mathfrak{S}_n$-Specht module corresponding to $\vec{c}$.

Their approach is to first relate $\Pi^\bullet_{\vec{c}}$ to a selected subcomplex $\Delta_{\vec{c}}$ of the simplicial complex $\Delta^\bullet_n$ of ordered set partitions of $\{1, 2, \ldots, n\}$ with last block possibly empty. In particular, they use Quillen's fiber lemma to show that $\Delta(\Pi^\bullet_{\vec{c}} \setminus \{\hat{1}\})$ is homotopy equivalent to $\Delta_{\vec{c}}$. They then give an explicit basis for $\widetilde{H}_{\mathrm{top}}(\Delta_{\vec{c}})$ that identifies the top homology as a Specht module.

Ehrenborg and Jung recover the results of Calderbank, Hanlon and Robinson [13] and Wachs [33] by specializing to $\vec{c} = (d, \ldots, d, d-1)$.

Taking a geometric viewpoint, one can consider $\Delta^\bullet_n$ as the barycentric subdivision of a distinguished facet of the standard simplex having vertices labeled with $\{1, 2, \ldots, n, n+1\}$. As such, it carries an action of $\mathfrak{S}_n$ and is a balanced simplicial complex, with each $\Delta_{\vec{c}}$ corresponding to a particular type-selected subcomplex. Under this identification, the poset $\Pi^\bullet_{\vec{c}}$ corresponds to linear subspaces spanned by faces in $\Delta_{\vec{c}}$.

We propose an analogous program for all well-generated complex reflection groups by introducing *well-framed* and *locally conical* systems. We complete the program for all irreducible *finite* groups having a presentation of the form

$$\langle r_1, \ldots, r_\ell \mid r_i^{p_i} = 1, \quad \underbrace{r_i r_j r_i \ldots}_{q_{ij}} = \underbrace{r_j r_i r_j \ldots}_{q_{ij}} \quad i < j \rangle \tag{1}$$

with $p_i \geq 2$ for all i and subject to the constraint that $p_i = p_j$ whenever q_{ij} is odd. Each such group has an irreducible faithful representation as a complex reflection group. The irreducible finite Coxeter groups are precisely those with each $p_i = 2$, i.e., those with a real form. The remaining groups are *Shephard groups*, the symmetry groups of regular complex polytopes.[1] The family of Coxeter and Shephard groups contains 21 of the 26 exceptional well-generated complex reflection groups. Using Shephard and Todd's numbering, the remaining five groups are $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$.

Outline

Solomon's ribbon/descent representations and their analogues appear naturally as homology representations within certain subcomplexes of a Coxeter-like complex $\Delta(W, R)$, and the idea of this paper is to transfer the ribbon/descent representations from $\Delta(W, R)$ to the lattice of intersections of reflecting hyperplanes for W using the map that takes each face to its linear span. The difficulty is in showing that the appropriate restrictions of this map do actually transfer the corresponding homology representations, in the strong sense that they define equivariant homotopy equivalences.

The first two sections set up a general framework for well-generated reflection groups. Roughly speaking, Section 2 focuses on $\Delta(W, R)$, and Section 3 focuses on the transfer. Section 2.1 gathers some preliminaries. Section 2.2 introduces *well-framed* and *strongly stratified* systems (W, R, Λ), then Sections 2.3–2.5 develop and connect the algebra and geometry of these systems. The vectors of the *frame* Λ play the role of fundamental weights for a Weyl group; the W-translates of the real hull of these vectors embed nicely inside unitary space to form a geometric realization of the abstract Coxeter-like complex $\Delta(W, R)$, whose faces are indexed by standard parabolic cosets.

Section 3.1 introduces the main objects of the paper: certain subcomplexes Δ^U_T of $\Delta(W, R)$ and certain subposets Π^U_T of the lattice of reflecting hyperplane intersections for W, where U and T are subsets of the set of generators R. The Δ^U_T's will naturally carry the ribbon/descent representations when $\Delta(W, R)$ is Cohen–Macaulay, and the Π^U_T's are the images of the Δ^U_T's under the map that takes a face to its linear span. Ehrenborg and Jung's pointed objects correspond to the special case

1 The algebraic unification of Coxeter groups and Shephard groups presented here does not appear to be widely known, and is attributed to Koster [21, p. 206].