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Nicely distance-balanced graphs

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ABSTRACT

A nonempty graph Γ is called *nicely distance-balanced*, whenever there exists a positive integer γ_Γ, such that for any two adjacent vertices u, v of Γ there are exactly γ_Γ vertices of Γ which are closer to u than to v, and exactly γ_Γ vertices of Γ which are closer to v than to u. The aim of this paper is to introduce the notion of nicely distance-balanced graphs, to provide examples of such graphs, to discuss relations between these graphs and distance-balanced and strongly distance-balanced graphs, as well as to prove some other results regarding these graphs.

1. Introduction

Let Γ be a finite, undirected, connected graph with diameter d, and let $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively. For $u, v \in V(\Gamma)$, we let $d(u, v) = d_\Gamma(u, v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of Γ we denote

$$W^\Gamma_{u,v} = W_{u,v} = \{x \in V(\Gamma) \mid d(x, u) < d(x, v)\}.$$

We say that Γ is *nicely distance-balanced* (NDB) whenever there exists a positive integer γ_Γ, such that for an arbitrary pair of adjacent vertices u and v of Γ

$$|W_{u,v}| = |W_{v,u}| = \gamma_\Gamma$$

holds. The notion of NDB graphs appears quite naturally in the context of distance-balanced and strongly distance-balanced graphs. Let us recall these definitions as well. We say that Γ is *distance-balanced* (DB) whenever for an arbitrary pair of adjacent vertices u and v of Γ

$$|W_{u,v}| = |W_{v,u}|$$

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holds. These graphs were, at least implicitly, first studied by Handa [6] who considered DB partial cubes. The term itself, however, is due to Jerebic, Klavžar and Rall [10] who studied DB graphs in the framework of various kinds of graph products.

Let uv be an arbitrary edge of Γ. For any two nonnegative integers i, j we let

$$D^i_j(u, v) = \{x \in V(\Gamma) \mid d(u, x) = i \text{ and } d(v, x) = j\}.$$

The triangle inequality implies that only the sets $D^{i-1}_i(u, v)$, $D^i_i(u, v)$ and $D^i_{i-1}(u, v)$ $(1 \le i \le d)$ can be nonempty. One can easily see that Γ is DB if and only if for every edge $uv \in E(\Gamma)$

$$\sum_{i=1}^{d} |D^i_{i-1}(u, v)| = \sum_{i=1}^{d} |D^{i-1}_i(u, v)| \tag{1}$$

holds.

Obviously, if $|D^i_{i-1}(u, v)| = |D^{i-1}_i(u, v)|$ holds for $1 \le i \le d$ and for every edge $uv \in E(X)$, then Γ is DB. The converse, however, is not necessarily true. For instance, in the generalized Petersen graphs GP(24, 4), GP(35, 8) and GP(35, 13) we can find two adjacent vertices u, v and an integer i, such that $|D^i_{i-1}(u, v)| \neq |D^{i-1}_i(u, v)|$. But it is easy to see that these graphs are DB. (The definition of the generalized Petersen graphs GP(n, k) is given in Section 3.)

We therefore say that Γ is *strongly distance-balanced* (SDB), if $|D^i_{i-1}(u, v)| = |D^{i-1}_i(u, v)|$ for every positive integer i and every edge $uv \in E(X)$. Let us remark that graphs with this property are also called *distance-degree regular* which were studied in [7].

For recent results on DB and SDB graphs see [1,4,8,13–16].

The aim of this paper is to introduce the notion of NDB graphs, to provide examples of such graphs, to discuss relations between NDB, DB and SDB property, as well as to prove some other results regarding these graphs. To do this, we first discuss some basic properties of NDB graphs in Section 2, and relations between NDB, DB and SDB graphs in Section 3. In Section 4 we study NDB graphs in the framework of various kinds of graph products. In Section 5 we classify NDB graphs Γ with $\gamma_\Gamma \in \{1, 2, 3\}$ and in Section 6 we study NDB graphs in the framework of graph symmetries.

2. Basic properties of NDB graphs

In this section we discuss some basic properties of NDB graphs regarding their diameter and their connection to edge-regular graphs. We start with the following easy observation.

Lemma 2.1. *Let Γ denote a connected NDB graph with n vertices and with diameter d. Then for every pair x, y of adjacent vertices of Γ, there are exactly $n - 2\gamma_\Gamma$ vertices of Γ, which are at the same distance from x and y. In other words,*

$$\sum_{i=1}^{d} |D^i_i(x, y)| = n - 2\gamma_\Gamma.$$

The following theorem shows that in a NDB graph Γ with diameter d we have $d \le \gamma_\Gamma$ and characterizes NDB graphs with $d = \gamma_\Gamma \ge 4$. The remaining two cases where $\gamma_\Gamma \in \{2, 3\}$ are considered in Section 5.

Theorem 2.2. *Let Γ be a NDB graph with diameter d. Then $d \le \gamma_\Gamma$. Moreover, if $d = \gamma_\Gamma \ge 4$ then Γ is either a $2d$-cycle or a $(2d + 1)$-cycle.*

Proof. Pick vertices x_0 and x_d of Γ such that $d(x_0, x_d) = d$ and a shortest path

$$x_0, x_1, x_2, \ldots, x_{d-1}, x_d$$

between x_0 and x_d. Note that $\{x_1, x_2, \ldots, x_d\} \subseteq W_{x_1,x_0}$. This shows that $d \le \gamma_\Gamma$.