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Critical groups of graphs with dihedral actions

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ABSTRACT

In this paper we consider the critical group of finite connected graphs which admit harmonic actions by the dihedral group D_n. In particular, we show that if the orbits of the D_n-action all have either n or $2n$ points then the critical group of such a graph can be decomposed in terms of the critical groups of the quotients of the graph by certain subgroups of the automorphism group. This is analogous to a theorem of Kani and Rosen which decomposes the Jacobians of algebraic curves with a D_n-action.

1. Introduction

In recent years, there have been a number of papers exploring the relationships between Jacobians of graphs (also known as critical groups) and Jacobians of algebraic curves. This connection was originally written about by Baker and Norine in [2] and has subsequently been explored in papers such as [3,4,7,8,12]. While there is not a literal dictionary between the two situations, several authors have shown that variations of theorems in algebraic geometry such as the Riemann–Hurwitz formula, the Hurwitz bound on the size of automorphism groups, and the Riemann–Roch theorem work in the setting of graph theory. At the same time, it is well known that the order of the critical group of a graph is the number of spanning trees on the graph, and recent papers such as [6,13,14] have given some results on the number of spanning trees of graphs which admit certain automorphisms.

Kani and Rosen proved the following result, which is actually a special case of [10, Theorem B], describing the relationship between the Jacobian of a curve which admits certain automorphisms and the Jacobians of the quotients by these automorphisms.

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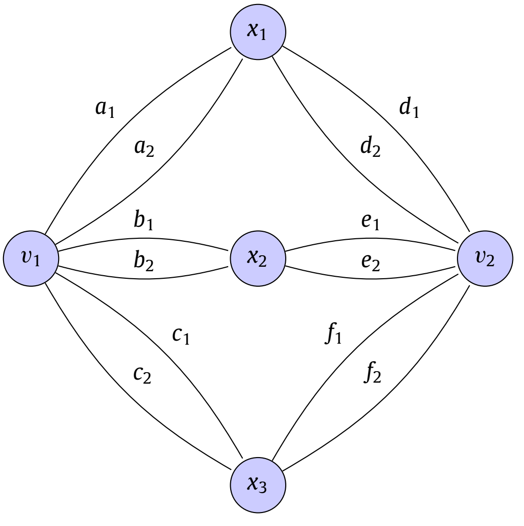
Theorem 1.1. *Let X be an algebraic curve so that the dihedral group D_n generated by two involutions σ_1 and σ_2 acts on X. Then we have the following isogeny relationship:*

$$\mathrm{Jac}(X) \oplus (\mathrm{Jac}(X/D_n))^2 \sim \mathrm{Jac}(X/\langle\sigma_1\rangle) \oplus \mathrm{Jac}(X/\langle\sigma_2\rangle) \oplus \mathrm{Jac}(X/\langle\sigma_1\sigma_2\rangle).$$

In this paper, we explore whether an analogous theorem will hold for the Jacobians of graphs. In particular, given a graph $\mathcal{G}$ with a dihedral action generated by two involutions σ_1 and σ_2, we investigate whether it is the case that the direct sum $\mathrm{Jac}(X/\langle\sigma_1\rangle) \oplus \mathrm{Jac}(X/\langle\sigma_2\rangle) \oplus \mathrm{Jac}(X/\langle\sigma_1\sigma_2\rangle)$ is a subgroup of $\mathrm{Jac}(X) \oplus (\mathrm{Jac}(X/D_n))^2$. The following example shows that, in general, this will not be the case.

Example 1.2. Consider the following graph $\mathcal{G}$:



Define σ_1 to be the involution on this graph which fixes the vertices v_1 and v_2 and permutes the edges by the map $(a_1a_2)(b_1c_2)(b_2c_1)(d_1d_2)(e_1f_2)(e_2f_1)$. Similarly, σ_2 will be the involution which fixes v_1 and v_2 and permutes the edges by the map $(a_1b_2)(a_2b_1)(c_1c_2)(d_1e_2)(d_2e_1)(f_1f_2)$. One can check that $\sigma_1\sigma_2$ has order three and therefore that the group generated by σ_1 and σ_2 is isomorphic to D_3. Moreover, the action of D_3 on G is harmonic and the graph G/D_3 is a tree.

It is straightforward to write down the Laplacian matrices of this graph and the quotient graphs and use the Smith Normal Forms to compute that

$$K(G/\langle\sigma_1\rangle) \cong K(G/\langle\sigma_2\rangle) \cong \mathbb{Z}/12\mathbb{Z}$$
$$K(G/\langle\sigma_1\sigma_2\rangle) \cong (\mathbb{Z}/2\mathbb{Z})^2$$
$$K(G) \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/12\mathbb{Z})$$

and in particular we conclude that $K(G/\langle\sigma_1\rangle) \oplus K(G/\langle\sigma_2\rangle) \oplus K(G/\langle\sigma_1\sigma_2\rangle)$ is not a subgroup of $K(G)$.

We suspect that the problem with this example is that every element of the group D_3 fixes the two vertices at either end, and in particular the inertia groups at these vertices is all of D_3. This is a situation that cannot occur for algebraic curves, where inertia groups must be cyclic except in the presence of wild ramification. While we are unable to prove that the theorem fully holds for all cyclic inertia groups, in this note we will show that a theorem along the lines of Theorem 1.1 holds in the case where all of the inertia groups are either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

More precisely, Theorems 5.3 and 4.4 will allow us to write down a decomposition of the group $K(G)$ in terms of the critical groups of the quotient graphs. In general, it appears that these decompositions do not split, but under slightly stronger hypothesis, we can prove the following direct analogue to Theorem 1.1.