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Countable locally 2-arc-transitive bipartite graphs

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ABSTRACT

We present an order-theoretic approach to the study of countably infinite locally 2-arc-transitive bipartite graphs. Our approach is motivated by techniques developed by Warren and others during the study of cycle-free partial orders. We give several new families of previously unknown countably infinite locally-2-arc-transitive graphs, each family containing continuum many members. These examples are obtained by gluing together copies of incidence graphs of semilinear spaces, satisfying a certain symmetry property, in a tree-like way. In one case we show how the classification problem for that family relates to the problem of determining a certain family of highly arc-transitive digraphs. Numerous illustrative examples are given.

1. Introduction

The study of finite graphs satisfying certain transitivity properties has a long and distinguished history with much of the motivation coming from the world of permutation groups. Various symmetry conditions have been considered, in various contexts, the more important among them including k-arc-transitivity (see [1] and Tutte's seminal work [28,29]), distance-transitivity (and distance-regularity) (see [2]), and homogeneity (see [14]).

The theory of infinite graphs satisfying transitivity conditions is in some aspects less developed, and necessarily so, since the powerful tools of finite group theory (like the classification of finite simple groups and the O'Nan–Scott Theorem for quasi-primitive groups) are not available to us. In this area of groups acting on infinite graphs (and other infinite relational structures) techniques and ideas

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from model theory play a key role. Arising from this, the strongest condition that has received attention, called *homogeneity*, has led to a classification for graphs (see [14,22]) and for other structures such as posets and digraphs [26,5]. However, for the other conditions mentioned above, even in instances where the picture for finite graphs is fairly complete, the situation for infinite graphs remains mysterious.

The objects of study in this paper will be the countable infinite 2-*arc-transitive graphs* and more generally *locally* 2-*arc-transitive graphs*. Given a graph $\Gamma = (V\Gamma, E\Gamma)$ with vertex set $V\Gamma$ and edge set $E\Gamma$, an *s-arc* in Γ is an $(s+1)$-tuple $v_0, v_1, \ldots, v_s$ of vertices such that v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}$ for all i. The graph Γ is *locally s-arc-transitive* if, with $G = \mathrm{Aut}(\Gamma)$, for each vertex v, the vertex stabilizer G_v is transitive on the set of s-arcs starting at v. If G is transitive on the set of all s-arcs in Γ we say that Γ is *s-arc-transitive*. Being s-arc-transitive is equivalent to being simultaneously both vertex transitive and locally s-arc-transitive. Interest in (finite) s-arc-transitive graphs goes back to the fundamental work of Tutte [28,29] showing that s-arc-transitive graphs of valency 3 satisfy $s \leq 5$. Later Weiss [32] proved that if the valency is at least 3, then $s \leq 7$. His result relies on the classification of finite simple groups, as do many other results in the area; see for example [23]. We remark that a graph is s-arc-transitive or locally s-arc transitive if and only if all its connected components are (and for s-arc-transitivity are all isomorphic), so we generally restrict to the case of connected graphs without further comment.

Let Γ be a connected s-arc-transitive graph, and $G = \mathrm{Aut}(\Gamma)$. If G does not act transitively on vertices and all vertices have valency at least two then, since G is edge-transitive, it follows that Γ is bipartite and G has two orbits on Γ, which are precisely the blocks of the bipartition of Γ. The locally s-arc-transitive graphs have received a lot of attention in the literature partly due to their links with areas of mathematics such as generalized n-gons, groups with a (B, N)-pair of rank two, Moufang polygons and Tutte's m-cages; see [32,7] for more details. Important recent papers about local s-arc-transitivity include [15–18]. In particular in [15] a programme of study of locally s-arc-transitive graphs for which G acts intransitively on vertices was initiated, with the O'Nan–Scott Theorem for quasi-primitive groups playing an important role in their reduction theorem. Interesting connections with semilinear spaces and homogeneous factorizations were explored in [17].

In this paper we shall consider countably infinite locally 2-arc-transitive bipartite graphs, which, by the comments above, include as a subclass the countably infinite locally 2-arc-transitive graphs for which G is intransitive on vertices. Our starting point is work of Droste [10] on *semilinear orders* and later of Warren (and others) on the, so called, 3-*CS*-transitive *cycle-free partial orders*. A surprising by-product of that work was that it led to a new and interesting family of countably infinite locally 2-arc-transitive bipartite graphs, essentially constructed by gluing linear orders together in a prescribed way (see Section 2 for a detailed description of this construction). This family of examples is of particular significance since it has no obvious finite analogue, that is, they are not simply examples that arise by taking some known finite family and allowing a parameter to go to infinity. Motivated by this, recent attempts have been made to see to what extent the methods of Warren may be used to start to try and understand countably infinite locally 2-arc-transitive bipartite graphs. Some recent work relating to these attempts includes [20,19]. However, until now the results have been fairly negative, highlighting some of the difficulties involved in generalizing Warren's techniques to obtain bipartite graphs with the required level of transitivity. In this article we succeed in an extension of the methods that does give (many) new examples of countably infinite locally 2-arc-transitive bipartite graphs, and in the process we discover some interesting connections with semilinear spaces (satisfying certain point–line–point transitivity properties) and the highly arc-transitive digraphs of Cameron et al. [3].

The paper is structured in the following way. In Section 2 we give the basic definitions and ideas required for the rest of the paper, and we explain our approach to the study of infinite locally 2-arc-transitive bipartite graphs. The idea is to reduce the analysis of such bipartite graphs to that of families determined by a poset derived from the bipartite graph, which we call its *interval*. We go on in Section 3 to look at one of these families in detail, namely the one in which the intervals are totally ordered, and the main results and constructions of the paper are given in that section. We concentrate on presenting a construction initially based on the integers, which is generalized to many of the countable 1-transitive linear orders in Morel's list [25], as well as one '2-coloured' case. As we shall explain, the general case gives rise to many more complications, even under the restrictions