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A combinatorial proof and refinement of a partition identity of Siladić

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ABSTRACT

In this paper we give a combinatorial proof and refinement of a Rogers–Ramanujan type partition identity of Siladić arising from the study of Lie algebras. Our proof uses q-difference equations.

1. Introduction

A partition of n is a non-increasing sequence of natural numbers whose sum is n. For example, there are 5 partitions of 4: 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. The Rogers–Ramanujan identities [13], first discovered by Rogers in 1894 and rediscovered by Ramanujan in 1917 are the following q-series identities:

Theorem 1.1. *Let $a = 0$ or 1. Then*

$$\sum_{k=0}^{n} \frac{q^{n(n+a)}}{(1-q)(1-q^2)\dots(1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+a+1})(1-q^{5k+4-a})}.$$

These analytic identities can be interpreted in terms of partitions in the following way:

Theorem 1.2. *Let $a = 0$ or 1. Then for every natural number n, the number of partitions of n such that the difference between two consecutive parts is at least 2 and the part 1 appears at most $1-a$ times is equal to the number of partitions of n into parts congruent to $\pm(1+a)$ mod 5.*

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Rogers–Ramanujan type partition identities establish equalities between certain types of partitions with difference conditions and partitions whose generating function is an infinite product.

Since the 1980s, many connections between representations of Lie algebras, q-difference equations and Rogers–Ramanujan type partition identities have emerged. For q-difference equations, see [6,8,9]. Regarding partitions, Lepowsky and Wilson [10] were the first to establish this link by giving an interpretation of Theorem 1.1 in terms of representations of the affine Lie algebra $sl_2(\mathbb{C})^{\sim}$. Similar methods were subsequently applied to other representations of affine Lie algebras, yielding new partition identities of the Rogers–Ramanujan type discovered by Capparelli [4], Primc [12] and Meurman–Primc [11] for example. Capparelli's conjecture was proved combinatorially by Alladi, Andrews and Gordon in [1] and Andrews in [2] just before Capparelli finished proving them with vertex-algebraic techniques [5]. Simultaneously, Tamba–Xie also proved Capparelli's conjecture using vertex operator theory [15]. However, many of the Rogers–Ramanujan type partition identities arising from the study of Lie algebras have yet to be understood combinatorially.

In [14], Siladić proved the following theorem by studying representations of the twisted affine Lie algebra $A_2^{(2)}$.

Theorem 1.3. *The number of partitions $\lambda_1 + \cdots + \lambda_s$ of an integer n into parts different from 2 such that difference between two consecutive parts is at least 5 (i.e. $\lambda_i - \lambda_{i+1} \geq 5$) and*

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 1, \pm 5, \pm 7 \bmod 16,$$
$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 2, \pm 6 \bmod 16,$$
$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 3 \bmod 16,$$
$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i + \lambda_{i+1} \not\equiv \pm 4 \bmod 16,$$

is equal to the number of partitions of n into distinct odd parts.

This paper is devoted to proving combinatorially and refining Theorem 1.3. In Section 2 we give an equivalent formulation of Theorem 1.3 which is easier to manipulate in terms of partitions. In Section 3 we establish q-difference equations satisfied by the generating functions of partitions considered in Theorem 1.3. Finally, we use those q-difference equations to prove Theorem 1.3 by induction.

Our refinement of Theorem 1.3 is the following:

Theorem 1.4. *For $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, let $A(k, n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that k equals the number of odd part plus twice the number of even parts, satisfying the following conditions:*

(1) $\forall i \geq 1, \lambda_i \neq 2$,
(2) $\forall i \geq 1, \lambda_i - \lambda_{i+1} \geq 5$,
(3) $\forall i \geq 1$,

$$\lambda_i - \lambda_{i+1} = 5 \Rightarrow \lambda_i \equiv 1, 4 \bmod 8,$$
$$\lambda_i - \lambda_{i+1} = 6 \Rightarrow \lambda_i \equiv 1, 3, 5, 7 \bmod 8,$$
$$\lambda_i - \lambda_{i+1} = 7 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 6, 7 \bmod 8,$$
$$\lambda_i - \lambda_{i+1} = 8 \Rightarrow \lambda_i \equiv 0, 1, 3, 4, 5, 7 \bmod 8.$$

For $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, let $B(k, n)$ denote the number of partitions of n into k distinct odd parts. Then for all $n \in \mathbb{N}$ and $k \in \mathbb{N}^*$, $A(k, n) = B(k, n)$.

2. Reformulating the problem

Our idea is to find q-difference equations and use them to prove Theorem 1.3, but its original formulation is not very convenient to manipulate combinatorially because it gives conditions on the sum of two consecutive parts of the partition. Therefore we will transform those conditions into conditions that only involve one part at a time.