European Journal of Combinatorics 36 (2014) 282–293

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Characterising planar Cayley graphs and Cayley complexes in terms of group presentations

Agelos Georgakopoulos [1,2]

Mathematics Institute, University of Warwick, CV4 7AL, UK

ARTICLE INFO

Article history:
Received 5 July 2012
Accepted 20 July 2013
Available online 22 August 2013

ABSTRACT

We prove that a Cayley graph can be embedded in the Euclidean plane without accumulation points of vertices if and only if it is the 1-skeleton of a Cayley complex that can be embedded in the plane after removing redundant simplices. We also give a characterisation of these Cayley graphs in term of group presentations, and deduce that they can be effectively enumerated.

1. Introduction

The study of groups that have Cayley graphs embeddable in the Euclidean plane $\mathbb{R}^2$, called *planar groups*, has a tradition starting in 1896 with Maschke's characterisation of the finite ones. Among the infinite planar groups, those that admit a *flat* Cayley complex, defined below, have received a lot of attention. They are important in complex analysis as they include the discontinuous groups of motions of the Euclidean and hyperbolic plane. Moreover, they are closely related to surface groups [19, Section 4.10]. These groups are now well understood due to the work of Macbeath [15], Wilkie [18], and others; see [19] for a survey.[3] Planar groups that have no flat Cayley complex are harder to analyse, and they are the subject of on-going research [4–6,8,7].

All groups, Cayley graphs and Cayley complexes in this paper are finitely generated. Our first result is

Theorem 1.1. *A planar Cayley graph of a group Γ is accumulation-free if and only if it is the* 1*-skeleton of a flat Cayley complex of Γ.*

[1] Tel.: +44 2476150900.

[2] Partly supported by FWF grant P-19115-N18.

[3] In [19] the term *Cayley complex* is not used but it is implicit in Theorems 4.5.6 and 6.4.7 that a group admits a flat Cayley complex if and only if it is a *planar discontinuous group*.

0195-6698/$ – see front matter
http://dx.doi.org/10.1016/j.ejc.2013.07.014

Here, a Cayley complex is *flat* if it can be embedded in $\mathbb{R}^2$ after removing redundant 2-simplices; see Section 2.1 for the precise definition. A planar graph is said to be *accumulation-free*, if it admits an embedding in $\mathbb{R}^2$ such that the images of its vertices have no accumulation point. The study of a planar graph is often simplified if one knows that the graph is accumulation-free; examples range from structural graph-theory [2] to percolation theory [14] and the study of spectral properties [13]. A further example is Thomassen's Theorem 5.2 below, which becomes false in the non-accumulation-free case. accumulation-free graphs can be characterised by a condition similar to that of Kuratowski's; see [9]. *Accumulation-free* embeddings also appear with other names in the literature, most notably "*locally finite*".

Theorem 1.1 implies that a group has a flat Cayley complex if and only if it has an accumulation-free Cayley graph, a fact that might be known to experts, and it should not be too hard to derive it from the results of [19]. Theorem 1.1 however strengthens this assertion into a theorem about all planar Cayley graphs, not just their groups. Since a single group can have a large variety of planar Cayley graphs (see Section 4 for some examples), it is in principle harder to prove results that hold for all planar Cayley graphs than proving the corresponding result for their groups. However, our proof is elementary and self-contained, avoiding the geometric machinery of [19].

We also prove that every accumulation-free Cayley graph admits an embedding the facial walks of which are preserved by the action of the group; see Corollary 3.6.

Finally, we derive a further characterisation of the accumulation-free Cayley graphs, and so by Theorem 1.1 also of the groups that admit a flat Cayley complex, by means of group presentations. We introduce a special kind of presentation, called a *facial presentation*, which is motivated by geometric intuition and can be easily recognised by an algorithm, and use it to obtain a further characterisation of the class of accumulation-free Cayley graphs:

Corollary 1.2. *A Cayley graph admits an accumulation-free embedding if and only if it admits a facial presentation.*

This implies that the accumulation-free Cayley graphs can be effectively enumerated (Corollary 5.4).

We prove Theorem 1.1 in Section 3. In Section 4 we examine accumulation-freeness as a group-theoretical invariant. Finally, in Section 5 we introduce facial presentations and prove Corollary 1.2.

2. Preliminaries

We will follow the terminology of [3] for graph-theoretical terms and that of [1,10] for group-theoretical ones.

Let us recall some standard definitions used in this paper. We say that a graph G is *k-connected* if $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. A *component* of G is a maximal connected subgraph of G.

A *walk* in G is an alternating sequence $v_0e_0v_1e_1 \cdots e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. If the vertices in a walk are all distinct, it is called a *path* (many authors use the word 'path' to denote a walk in our sense).

A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*. Two rays contained in a graph G are *equivalent* if no finite set of edges separates them. The corresponding equivalence classes of rays are the *ends* of G.

By an *embedding* of a graph G we mean a topological embedding of the corresponding 1-complex in the Euclidean plane $\mathbb{R}^2$; in simpler words, an embedding is a drawing of the graph in the plane with no two edges crossing. A graph is *planar* if it admits an embedding. A *plane* graph is a (planar) graph endowed with a fixed embedding.

A *face* of an embedding $\sigma : G \to \mathbb{R}^2$ is a component of $\mathbb{R}^2 \setminus \sigma(G)$. The *boundary* of a face F is the set of vertices and edges of G that are mapped by σ to the closure of F. A path, or walk, in G is called *facial* with respect to σ if it is contained in the boundary of some face of σ.

One of our main tools will be the (finitary) *cycle space* $\mathcal{C}_f(G)$ of a graph $G = (V, E)$, which is defined as the vector space over $\mathbb{Z}_2$ (the field of two elements) consisting of those subsets of E such that can