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Deformation of the Hopf algebra of plane posets

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ABSTRACT

We describe and study a four parameters deformation of the two products and the coproduct of the Hopf algebra of plane posets. We obtain a family of braided Hopf algebras, which are generically self-dual. We also prove that in a particular case (when the second parameter goes to zero and the first and third parameters are equal), this deformation is isomorphic, as a self-dual braided Hopf algebra, to a deformation of the Hopf algebra of free quasi-symmetric functions **FQSym**.

0. Introduction

A *double poset* is a finite set with two partial orders. As explained in [12], the space generated by the double posets inherits two products and one coproduct, here denoted by $\rightsquigarrow$, $\lightning$ and Δ, making it both a Hopf and an infinitesimal Hopf algebra [10]. Moreover, this Hopf algebra is self dual. When the second order is total, we obtain the notion of special posets, also called labelled posets [13] or shapes [1]. A double poset is *plane* if its two partial orders satisfy a (in)compatibility condition, see Definition 1. The subspace $\mathcal{H}_{\mathcal{PP}}$ generated by plane posets is stable under the two products and the coproduct, and is self-dual as a Hopf algebra [3,4]: in particular, two Hopf pairings are defined on it, using the notion of *picture* [6,8,9,14]. Moreover, as proved in [4], it is isomorphic to the Hopf algebra of free quasi-symmetric functions **FQSym**, also known as the Malvenuto–Reutenauer Hopf algebra of permutations. An explicit isomorphism Θ is given by the linear extensions of plane posets, see Definition 11.

We define in this text a four parameters deformation of the products and the coproduct of $\mathcal{H}_{\mathcal{PP}}$, together with a deformation of the two pairings and of the morphism from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym**. If $q = (q_1, q_2, q_3, q_4) \in K^4$, the product $m_q(P \otimes Q)$ of two plane posets P and Q is a linear span of plane posets R such that $R = P \sqcup Q$ as a set, P and Q being plane subposets of R. The coefficients are defined

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with the help of the two partial orders of R, see Theorem 14, and are polynomials in q. In particular:

$$\begin{cases} m_{(1,0,0,0)} = \not\downarrow, \\ m_{(0,1,0,0)} = \not\downarrow^{op}, \\ m_{(0,0,1,0)} = \rightsquigarrow, \\ m_{(0,0,0,1)} = \rightsquigarrow^{op}. \end{cases}$$

We also obtain the product dual to the coproduct Δ (considering the basis of double posets as orthonormal) as $m_{(1,0,1,1)}$, and its opposite given by $m_{(0,1,1,1)}$. Dually, we define a family of coassociative coproducts Δ_q. For any plane poset P, $\Delta(P)$ is a linear span of terms $(P \setminus I) \otimes I$, running over the plane subposets I of P, the coefficients being polynomials in q. In the particular cases where at least one of the components of q is zero, only h-ideals, r-ideals or biideals can appear in this sum (Definition 7 and Proposition 22). We study the compatibility of Δ_q with both products $\rightsquigarrow$ and $\not\downarrow$ on $\mathcal{H}_{\mathcal{PP}}$ (Proposition 23). In particular, $(\mathcal{H}_{\mathcal{PP}}, \rightsquigarrow, \Delta_q)$ satisfies the axiom

$$\Delta_q(x \rightsquigarrow y) = \sum \sum q_3^{|x'_q||y''_q|} q_4^{|x''_q||y'_q|} (x'_q \rightsquigarrow y'_q) \otimes (x''_q \rightsquigarrow y''_q).$$

If $q_3 = 1$, it is a braided Hopf algebra, with the braiding given by $c_q(P \otimes Q) = q_4^{|P||Q|} Q \otimes P$; if $q_3 = 1$ and $q_4 = 1$, this is a Hopf algebra, and if $q_3 = 1$ and $q_4 = 0$ this is an infinitesimal Hopf algebra. If $q_4 = 1$, this is the co-opposite (or the opposite) of a braided Hopf algebra. Similar results hold if we consider the second product $\not\downarrow$, permuting the roles of (q_3, q_4) and (q_1, q_2).

We define a symmetric pairing $\langle -, - \rangle_q$ such that:

$$\langle x \otimes y, \Delta_q(z) \rangle_q = \langle x \rightsquigarrow y, z \rangle_q \text{ for all } x, y, z \in \mathcal{H}_{\mathcal{PP}}.$$

If $q = (1, 0, 1, 1)$, we recover the first "classical" pairing of $\mathcal{H}_{\mathcal{PP}}$. We prove that in the case $q_2 = 0$, this pairing is nondegenerate if, and only if, $q_1 \neq 0$ (Corollary 36). Consequently, this pairing is generically nondegenerate.

The coproduct of **FQSym** is finally deformed, in such a way that the algebra morphism Θ from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym** becomes compatible with Δ_q, if q has the form $q = (q_1, 0, q_1, q_4)$. Deforming the second pairing $\langle -, - \rangle'$ of $\mathcal{H}_{\mathcal{PP}}$ and the usual Hopf pairing of **FQSym**, the map Θ becomes also an isometry (Theorem 40). Consequently, the deformation $\langle -, - \rangle'_q$ is nondegenerate if, and only if, $q_1 q_4 \neq 0$.

This text is organized as follows. The first section contains reminders on the Hopf algebra of plane posets $\mathcal{H}_{\mathcal{PP}}$, its two products, its coproducts and its two Hopf pairings, and on the isomorphism Θ from $\mathcal{H}_{\mathcal{PP}}$ to **FQSym**. The deformation of the products is defined in Section 2, and we also consider the compatibility of these products with several bijections on $\mathcal{PP}$ and the stability of certain families of plane posets under these products. We proceed to the dual construction in the next section, where we also study the compatibility with the two (undeformed) products. The deformation of the first pairing is described in Section 4. The compatibilities with the bijections on $\mathcal{PP}$ or with the second product $\not\downarrow$ are also given, and the nondegeneracy is proved for $q = (q_1, 0, q_3, q_4)$ if $q_1 \neq 0$. The last section is devoted to the deformation of the second pairing and of the morphism to **FQSym**.

1. Backgrounds and notations

1.1. Double and plane posets

Definition 1. 1. [12] A *double poset* is a triple $(P, \leq_1, \leq_2)$, where P is a finite set and $\leq_1, \leq_2$ are two partial orders on P.

2. A *plane poset* is a double poset $(P, \leq_h, \leq_r)$ such that for all $x, y \in P$, such that $x \neq y$, x and y are comparable for $\leq_h$ if, and only if, x and y are not comparable for $\leq_r$. The set of (isoclasses of) plane posets will be denoted by $\mathcal{PP}$. For all $n \in \mathbb{N}$, the set of (isoclasses of) plane posets of cardinality n will be denoted by $\mathcal{PP}(n)$.

Examples. Here are the plane posets of cardinal ≤ 4. They are given by the Hasse graph of $\leq_h$; if x and y are two vertices of this graph which are not comparable for $\leq_h$, then $x \leq_r y$ if y is more on the right