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Indecomposability graph and indecomposability recognition☆

A. Boussaïri[a], A. Chaïchaâ[a], P. Ille[b]

[a] Faculté des Sciences Aïn Chock, Département de Mathématiques et Informatique, Km 8 route d'El Jadida, BP 5366 Maarif, Casablanca, Maroc

[b] Institut de Mathématiques de Luminy, CNRS – UMR 6206, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 09, France

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ABSTRACT

Given a digraph $G = (V, A)$, a subset X of V is an interval of G if for $a, b \in X$ and $v \in V \setminus X$, $(a, v) \in A$ if and only if $(b, v) \in A$, and similarly for (v, a) and (v, b). For instance, $\emptyset$, V and $\{v\}$, $v \in V$, are intervals of G called trivial. A digraph is indecomposable if all its intervals are trivial. Let $G = (V, A)$ be a digraph. Given $v \in V$, v is an indecomposability vertex of G if $G[V \setminus \{v\}]$ is indecomposable. The indecomposability graph $\mathbb{I}(G)$ of G is defined on V as follows. Given $v \neq w \in V$, $\{v, w\}$ is an edge of $\mathbb{I}(G)$ if $G[V \setminus \{v, w\}]$ is indecomposable. The following is proved for an indecomposable digraph $G = (V, A)$. For every digraph $H = (V, B)$, if G and H have the same indecomposability vertices and if $d_{\mathbb{I}(G)}(v) = d_{\mathbb{I}(H)}(v)$ for each $v \in V$, then H is indecomposable. We also study other types of indecomposability recognition.

1. Introduction

A *digraph* G consists of a finite and nonempty *vertex* set $V(G)$ and an *arc* set $A(G)$, where an arc is an ordered pair of distinct vertices. Such a digraph is denoted by $(V(G), A(G))$ or simply by (V, A). For example, given a finite and nonempty set V, $(V, (V \times V) \setminus \{(v, v) : v \in V\})$ is the *complete* digraph on V. Given a digraph $G = (V, A)$, with each nonempty subset X of V, we associate the *subdigraph* $G[X] = (X, A \cap (X \times X))$ of G induced by X. Given a proper subset X of V, $G[V \setminus X]$ is also denoted by $G - X$, and by $G - v$ whenever $X = \{v\}$. Two interesting types of digraphs are tournaments and

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E-mail addresses: aboussairi@hotmail.com (A. Boussaïri), chaichaa@hotmail.com (A. Chaïchaâ), ille@iml.univ-mrs.fr (P. Ille).

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symmetric digraphs. A digraph G is a *tournament* if $|A(G) \cap \{(v, w), (w, v)\}| = 1$ for any $v \neq w \in V$. On the other hand, a digraph G is *symmetric* if $|A(G) \cap \{(v, w), (w, v)\}| = 0$ or 2 for any $v \neq w \in V$. A *graph* G is defined by a finite and nonempty *vertex* set $V(G)$ and an *edge* set $E(G)$, where an edge is an unordered pair of distinct vertices. Such a graph is denoted by $(V(G), E(G))$ or simply by (V, E). For example, given a finite and nonempty set V, $(V, \emptyset)$ is the *empty* graph on V whereas $(V, \binom{V}{2})$ is the *complete* graph. With a partition $\{V_1, V_2\}$ of V, where $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, we associate the *complete bipartite* graph $(V, \{\{v_1, v_2\} : v_1 \in V_1, v_2 \in V_2\})$. Let G be a graph. Given $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v in G is the family of vertices w of G such that $\{v, w\} \in E(G)$. The cardinality of $N_G(v)$ is called the *degree* of v and is denoted by $d_G(v)$. We say that v is an *isolated* vertex of G whenever $d_G(v) = 0$. With each nonempty subset X of $V(G)$, we associate the *subgraph* $G[X] = (X, E(G) \cap \binom{X}{2})$ of G induced by X. A nonempty subset C of $V(G)$ is a *connected component* of G if for any $x \in C$ and $y \in V(G) \setminus C$, $\{x, y\} \notin E(G)$ and if for any $x \neq y \in C$, there is a sequence $x_0 = x, \ldots, x_n = y$ of elements of C such that $\{x_i, x_{i+1}\} \in E(G)$ for $0 \leq i \leq n - 1$.

Given digraphs G and H, a bijection f from $V(G)$ onto $V(H)$ is an *isomorphism* from G onto H provided that for any $x, y \in V(G)$, $(x, y) \in A(G)$ if and only if $(f(x), f(y)) \in A(H)$. Two digraphs are then *isomorphic* if there exists an isomorphism from one onto the other. Let G be a digraph. For $x \neq y \in V(G)$ and for $u \neq v \in V(G)$, we write $[x, y] \equiv [u, v]$ if the function $\{x, y\} \longrightarrow \{u, v\}$, defined by $x \mapsto u$ and $y \mapsto v$, is an isomorphism from $G[\{x, y\}]$ onto $G[\{u, v\}]$; otherwise, we write $[x, y] \not\equiv [u, v]$.

Given a digraph G, a subset I of $V(G)$ is an *interval* [3,8] (or an *autonomous* set [5,9,12] or a *clan* [4] or a *homogeneous* set [2,6] or a *module* [15]) of G provided that for any $a, b \in I$ and $v \in V(G) \setminus I$, we have $[a, v] \equiv [b, v]$. For instance, $\emptyset$, $V(G)$ and $\{v\}$, where $v \in V(G)$, are intervals of G called *trivial* intervals. A digraph is *indecomposable* [3,8,14] (or *prime* [2] or *primitive* [4]) if all its intervals are trivial; otherwise, it is *decomposable*. Given a digraph G, an *indecomposability* vertex of G is an element v of $V(G)$ such that $G - v$ is indecomposable. We denote by $\mathcal{I}(G)$ the set of the indecomposability vertices of G. An indecomposable digraph is *critical* if it does not admit indecomposability vertices. Critical and indecomposable digraphs were characterized by Schmerl and Trotter [14].

Given two digraphs $G = (V, A)$ and $H = (V, B)$, we consider an integer k such that $0 < k < |V|$. We say that G and H are *$\{-k\}$-hypomorphic* if $G - X$ and $H - X$ are isomorphic for every subset X of V with $|X| = k$. A digraph G is *$\{-k\}$-reconstructible* if any digraph *$\{-k\}$-hypomorphic* to G is isomorphic to it. Ulam [17] conjectured that a symmetric digraph with at least three vertices is $\{-1\}$-reconstructible. Then, Pouzet [1, Problem 24] asked whether a digraph is $\{-k\}$-reconstructible. It follows from results of Lopez and Rauzy [10,11] and of Pouzet [13] that a digraph is $\{-k\}$-reconstructible for $k \geq 4$. For $k = 1$, Stockmeyer [16] built an infinite family of non $\{-1\}$-reconstructible tournaments. However, all these tournaments are indecomposable. So the conjecture is still open for decomposable digraphs. Ille [7] showed that two $\{-1\}$-hypomorphic digraphs $G = (V, A)$ and $H = (V, B)$, with $|V| \geq 11$, are both indecomposable or not. He introduced the indecomposability graph as follows. Given a digraph G, the *indecomposability* graph of G is the graph $\mathbb{I}(G)$ defined on the set $V(G)$ in the following way. Given $v \neq w \in V$, $\{v, w\} \in E(\mathbb{I}(G))$ if $G - \{v, w\}$ is indecomposable. Clearly, two $\{-1\}$-hypomorphic digraphs share the same indecomposability vertices. However, for any finite and nonempty set V, there are digraphs $G = (V, A)$ and $H = (V, B)$ with the same indecomposability vertices, such that G is indecomposable and H is decomposable. It suffices to consider a critical and indecomposable digraph defined on V and the complete digraph on V. The above argumentation leads us to the following natural definition of the indecomposability recognition. First, two digraphs are said to be *equivalent under indecomposability* or simply *equivalent* if both are indecomposable or not. Second, given two digraphs $G = (V, A)$ and $H = (V, B)$, consider an integer k such that $0 < k < |V|$. We say that G and H are *$\{-k\}$-equivalent* if $G - X$ and $H - X$ are equivalent for every subset X of V with $|X| = k$. A digraph G is then said to be *$\{-k\}$-recognizable* if any digraph $\{-k\}$-equivalent to G is equivalent to it. Similarly, we define the $\mathcal{F}$-recognition of a digraph G for a family $\mathcal{F}$ of integers m such that $-|V(G)| < m < 0$. Notice that two digraphs G and H are $\{-1\}$-equivalent if and only if $\mathcal{I}(G) = \mathcal{I}(H)$. They are $\{-2\}$-equivalent if and only if $\mathbb{I}(G) = \mathbb{I}(H)$.

As observed above, critical and indecomposable digraphs are not $\{-1\}$-recognizable. In Section 4, we give a counter-example to $\{-2\}$-recognition. In Section 5, we study the indecomposability vertices