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Representing finite convex geometries by relatively convex sets

Kira Adaricheva

Yeshiva University, 245 Lexington Ave., New York, NY 10016, USA

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ABSTRACT

A closure system with the anti-exchange axiom is called a convex geometry. One geometry is called a sub-geometry of the other if its closed sets form a sublattice in the lattice of closed sets of the other. We prove that convex geometries of relatively convex sets in n-dimensional vector space and their finite sub-geometries satisfy the n-Carousel Rule, which is the strengthening of the n-Carathéodory property. We also find another property, that is similar to the simplex partition property and independent of 2-Carousel Rule, which holds in sub-geometries of 2-dimensional geometries of relatively convex sets.

1. Introduction

A closure system $\mathbf{A} = (A, ^{-})$, i.e. a set A with a closure operator $^{-} : 2^A \to 2^A$ defined on A, is called *a convex geometry* (see [3]), if it is a zero-closed space (i.e. $\overline{\emptyset} = \emptyset$) and it satisfies *the anti-exchange axiom*, i.e.

$$x \in \overline{X \cup \{y\}} \text{ and } x \notin X \text{ imply that } y \notin \overline{X \cup \{x\}} \text{ for all } x \neq y \text{ in } A \text{ and all closed } X \subseteq A.$$

A convex geometry $\mathbf{A} = (A, ^{-})$ is called finite, if set A is finite.

Very often, a convex geometry is given by its collection of closed sets. There is a convenient description of those collections of subsets of a given finite set A, which are, in fact, the closed sets of a convex geometry on A: if $\mathcal{F} \subseteq 2^A$ satisfies

(1) $\emptyset \in \mathcal{F}$;
(2) $X \cap Y \in \mathcal{F}$, as soon as $X, Y \in \mathcal{F}$;
(3) $X \in \mathcal{F}$ and $X \neq A$ implies $X \cup \{a\} \in \mathcal{F}$, for some $a \in A \setminus X$,

then $\mathcal{F}$ represents the collection of closed sets of a convex geometry $\mathbf{A} = (A, ^{-})$.

E-mail address: adariche@yu.edu.

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A reader can be referred to [8,9] for the further details of combinatorial and lattice-theoretical aspects of finite convex geometries.

For convex geometries $\mathbf{A} = (A, ^-)$ and $\mathbf{B} = (B, \tau)$, one says that $\mathbf{A}$ is a sub-geometry of $\mathbf{B}$, if there is a one-to-one map ϕ of closed sets of $\mathbf{A}$ to closed sets of $\mathbf{B}$ such that $\phi(X \cap Y) = \phi(X) \cap \phi(Y)$, and $\phi(\overline{X \cup Y}) = \tau(\phi(X) \cup \phi(Y))$, where $X, Y \subseteq A, \overline{X} = X, \overline{Y} = Y$. In other words, the lattice of $^-$-closed subsets of $\mathbf{A}$ is a sublattice of the lattice of τ-closed sets of $\mathbf{B}$. When geometries $\mathbf{A}$ and $\mathbf{B}$ are defined on the same set $X = A = B$, we also call $\mathbf{B}$ *a strong extension* of $\mathbf{A}$. Extensions of finite convex geometries were considered in [4,3], the more systematic treatment of extensions of finite lattices was given in [15].

Given any class $\mathcal{L}$ of convex geometries, we will call it *universal*, if an arbitrary finite convex geometry is a sub-geometry of some geometry in $\mathcal{L}$.

One of the main results in [3] proves that a specially designed class of convex geometries $\mathcal{AL}$ is universal. Namely, $\mathcal{AL}$ consists of convex geometries of the form $S_p(A)$, each of which is built on a carrier set of an algebraic and dually algebraic lattice A and whose closed sets are all complete lower subsemilattices of A closed with respect to taking joins of non-empty chains. At the same time, a subclass $\mathcal{AL}_f$ of all *finite* geometries in $\mathcal{AL}$ cease to be universal, see [1,3,17].

In this paper, we want to consider another conveniently designed class of convex geometries, in fact, an infinite hierarchy of classes.

Given a set of points A in Euclidean n-dimensional space $\mathbb{R}^n$, one defines a closure operator $^- : 2^A \to 2^A$ on A as follows: for any $Y \subseteq A, \overline{Y} = ch(Y) \cap A$, where ch stands for *the convex hull*. One easily verifies that such an operator satisfies the anti-exchange axiom. Thus, $(A, ^-)$ is a convex geometry, which also will be denoted as $\mathbf{Co}(\mathbb{R}^n, A)$. We will call such convex geometry *a geometry of relatively convex sets* (assuming that these are convex sets "relative" to A). The convex geometries of relatively convex sets were studied in [2,7,13,18].

For any geometry $\mathbf{C} = \mathbf{Co}(\mathbb{R}^m, A)$, call $n \in \mathbb{N}$ *a dimension* of $\mathbf{C}$, if n is the smallest number such that $\mathbf{C}$ could be represented as $\mathbf{Co}(\mathbb{R}^n, A)$, for appropriate $A \subseteq R^n$. In particular, $n \leq m$, and $n \leq p - 1$, if A is a finite non-empty set of cardinality $p > 1$.

Let $\mathcal{C}_n$ be the class of convex geometries of relatively convex sets of dimension $\leq n$. It is known that none of $\mathcal{C}_n$ is universal, due to the n-Carathéodory property that holds on any sub-geometry of geometry from $\mathcal{C}_n$ (see, for example, [7]), but fails on any geometry of dimension $n + 1$.

In Section 2, we introduce a stronger property called the n-Carousel Rule and show that it holds on sub-geometries of $\mathcal{C}_n$. It allows to build, in Section 3, a series of finite convex geometries $\mathbf{K}_n$ such that $\mathbf{K}_n$ satisfies the n-Carathéodory property, but cannot be a sub-geometry of any geometry in $\mathcal{C}_n$. On the other hand, $\mathbf{K}_n$ is a sub-geometry of some geometry in $\mathcal{C}_{n+1}$.

In Section 4 we also introduce the so-called *Edge Carousel Rule*, which is a slight modification of *the simplex partition property* from [14]. We prove that, similar to the 2-Carousel Rule, this property holds in all sub-geometries in $\mathcal{C}_2$, and we give examples to demonstrate that the Edge Carousel Rule is independent of the 2-Carousel Rule.

This raises the question of whether these two properties characterize finite sub-geometries of $\mathcal{C}_2$. More generally, we would like to find the description of finite sub-geometries of $\mathcal{C}_n$, for arbitrary n. This approach may be helpful in tackling a problem raised in [3]: whether every finite convex geometry is a sub-geometry of $\mathbf{C} = \mathbf{Co}(\mathbb{R}^m, A)$, for some m and *finite* A. We conclude the paper with Section 5, where we discuss the open problems with more detail.

2. The Carathéodory property and the Carousel Rule

We recall that a convex geometry $(A, ^-)$ satisfies the *n-Carathéodory property*, if $x \in \overline{S}, S \subseteq A$, implies $x \in \overline{\{a_0, \ldots, a_n\}}$ for some $a_0, \ldots, a_n \in S$. Equivalently, a_0 can be taken to be any pre-specified element of S: if $x \in \overline{S}, S \subseteq A$ and $a_0 \in S$, then $x \in \overline{\{a_0, \ldots, a_n\}}$ for some $a_1, \ldots, a_n \in S$.

Proposition 2.1 *([13, Lemma 3.2], [7, Proposition 25]). For any $n \in \mathbb{N}$ and $A \subseteq \mathbb{R}^n$, convex geometry $\mathbf{Co}(\mathbb{R}^n, A)$ satisfies the n-Carathéodory property.*

Our aim is to formulate a stronger property, which we call the *n-Carousel Rule*, extending to arbitrary finite dimensions the 2-Carousel Rule introduced in [5].