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Locally bounded coverings and factorial properties of graphs

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ABSTRACT

For a graph property X, let X_n be the number of graphs with vertex set $\{1, \ldots, n\}$ having property X, also known as the speed of X. A property X is called factorial if X is hereditary (i.e. closed under taking induced subgraphs) and $n^{c_1 n} \leq X_n \leq n^{c_2 n}$ for some constants c_1 and c_2. Hereditary properties with speed slower than factorial are surprisingly well structured. The situation with factorial properties is more complicated and less explored. Only the properties with speeds up to the Bell number are well studied and well behaved. To better understand the behavior of factorial properties with faster speeds we introduce a structural tool called *locally bounded coverings* and show that a variety of graph properties can be described by means of this tool.

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1. Introduction

A *graph property* is an infinite class of graphs closed under isomorphism. Given a property X, we write X_n for the number of graphs in X with vertex set $\{1, 2, \ldots, n\}$. Following [7], we call X_n the *speed* of the property X.

A property is *hereditary* if it is closed under taking induced subgraphs. It is well-known (and can be easily seen) that a graph property X is hereditary if and only if X can be described in terms of forbidden induced subgraphs. More formally, for a set of graphs M let us denote by $\text{Free}(M)$ the class of graphs containing no induced subgraphs isomorphic to graphs in the set M. Then X is a hereditary class if and only if $X = \text{Free}(M)$ for some set M. We call M the set of *forbidden induced subgraphs* for the class X and say that graphs in X are M-free.

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The speed of hereditary properties and their asymptotic structure have been extensively studied, originally in the special case of a single forbidden subgraph, and more recently in general. For example, Erdős et al. [14] and Kolaitis et al. [18] studied K_r-free graphs, Erdős et al. [13] studied properties where a single graph is forbidden as a subgraph (not necessarily induced), and Prömel and Steger obtained a number of results [21,23,22] for properties defined by a single forbidden induced subgraph. This line of research culminated in a breakthrough result stating that for every hereditary class X different from the class of all finite graphs,

$$\lim_{n\to\infty} \frac{\log_2 X_n}{\binom{n}{2}} = 1 - \frac{1}{k(X)}, \quad (1)$$

where $k(X)$ is a natural number, called the *index* of X. To define this notion let us denote by $\mathcal{E}_{i,j}$ the class of graphs whose vertices can be partitioned into at most i independent sets and j cliques. In particular, $\mathcal{E}_{2,0}$ is the class of bipartite graphs and $\mathcal{E}_{1,1}$ is the class of split graphs. Then $k(X)$ is the largest k such that X contains $\mathcal{E}_{i,j}$ with $i+j = k$. This result was obtained independently by Alekseev [2] and Bollobás and Thomason [11,12] and is known nowadays as the Alekseev–Bollobás–Thomason Theorem (see e.g. [6]).

Since $\binom{n}{2}$ is the minimum number of bits needed to represent an arbitrary n-vertex labeled graph and $\log_2 X_n$ is the minimum number of bits needed to represent an n-vertex labeled graph in the class X, the ratio $\log_2 X_n / \binom{n}{2}$ can be viewed as the coefficient of compressibility for representing graphs in X and its asymptotic value was called by Alekseev [1] the *entropy* of X.

In [1], Alekseev proposed an efficient algorithm which gives an asymptotically optimal coding for graphs in every hereditary class X of index $k > 1$. For classes of index $k = 1$, which were called *unitary*, this algorithm is not optimal, since the equality (1) does not provide the asymptotic behavior of $\log_2 X_n$.

The family of unitary classes (i.e. classes of index 1) contains many classes of theoretical or practical importance, such as line graphs, interval graphs, permutation graphs, threshold graphs, forests, chordal bipartite, planar graphs and, even more generally, all proper minor-closed graph classes [20], all classes of graphs of bounded vertex degree, of bounded clique-width [5], etc.

In order to differentiate the family of unitary classes in accordance with their speeds, Alekseev [3] proposed the following definition. Let us call two classes X and Y *isometric* if there are positive constants c_1, c_2 and n_0 such that $Y_n^{c_1} \le X_n \le Y_n^{c_2}$ for all $n \ge n_0$. It is not difficult to see that the isometricity is an equivalence relation, and the equivalence classes of this relation will be called *layers*.

All classes of index greater than 1 form a single layer and all classes containing finitely many graphs form a single layer. Between these two extremes lies the family of unitary classes and it consists of infinitely many layers. To see this, consider the class Z^p of bipartite graphs containing no $K_{p,p}$ as an induced subgraph. From the well-known results on the maximum number of edges in graphs in Z^p (see e.g. [10,15]), we have

$$c_1 n^{2-\frac{2}{p+1}} < \log_2 |Z_n^p| < c_2 n^{2-\frac{1}{p}} \log_2 n,$$

which implies, in particular, that Z^p and Z^{2p} are non-isometric.

The first four lower layers in the family of unitary classes have been distinguished by Scheinerman and Zito in [24]. These are:

- the *constant* layer contains classes X with $\log_2 X_n = O(1)$,
- the *polynomial* layer contains classes X with $\log_2 X_n = \Theta(\log_2 n)$,
- the *exponential* layer contains classes X with $\log_2 X_n = \Theta(n)$,
- the *factorial* layer contains classes X with $\log_2 X_n = \Theta(n \log_2 n)$.

Independently, similar results have been obtained by Alekseev in [3]. Moreover, Alekseev provided the first four layers with the description of all minimal classes, i.e. he identified in each layer a family of classes every hereditary subclass of which belongs to a lower layer (see also [7]). In particular, the factorial layer has nine minimal classes; three of these are subclasses of bipartite graphs, three others are subclasses of co-bipartite graphs and the remaining three are subclasses of split graphs. The three minimal factorial classes of bipartite graphs are:

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