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# Generic rectangulations

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ABSTRACT

A rectangulation is a tiling of a rectangle by a finite number of rectangles. The rectangulation is called generic if no four of its rectangles share a single corner. We initiate the enumeration of generic rectangulations up to combinatorial equivalence by establishing an explicit bijection between generic rectangulations and a set of permutations defined by a pattern-avoidance condition analogous to the definition of the twisted Baxter permutations.


## 1. Introduction

The main characters in this paper are tilings of a rectangle by finitely many rectangles. A *cross* in such a tiling is a point which is a corner of four distinct tiles. Fixing a rectangle $S$ and considering the space of all tilings of $S$ by $n$ rectangles with a uniform probability measure, the set of tilings having one or more crosses has measure zero. Thus we call a tiling *generic* if it has no crosses.

We consider generic tilings up to the natural combinatorial equivalence relation which we now describe. We orient $S$ so that its edges are vertical and horizontal. A rectangle $U$ in a tiling $R$ is *below* a rectangle $V$ if the top edge of $U$ intersects the bottom edge of $V$ (necessarily in a line segment rather than in a point). Similarly, $U$ is *left of* $V$ if the right edge of $U$ intersects the left edge of $V$. A tiling $R$ of a rectangle $S$ is combinatorially equivalent to a tiling $R'$ of a rectangle $S'$ if there is a bijection from the rectangles of $R$ to the rectangles of $R'$ that exactly preserves the relations "below" and "left of". A *generic rectangulation* is the equivalence class of a generic tiling. We will often blur the distinction between generic rectangulations (i.e. equivalence classes) and equivalence class representatives, in particular specifying an equivalence class by describing a specific tiling.

Our main result is a bijection between generic rectangulations with $n$ rectangles and a class of permutations in $S_n$ that we call 2-*clumped permutations*. These are the permutations that avoid the patterns 3-51-2-4, 3-51-4-2, 2-4-51-3, and 4-2-51-3, in the notation of Babson and Steingrímsson [5], which is explained in Section 2. The author's counts of generic rectangulations, for small $n$, are shown in Table 1.

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**Table 1**
The number of generic rectangulations with $n$ rectangles.

| $n$ | Generic rectangulations |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 6 |
| 4 | 24 |
| 5 | 116 |
| 6 | 642 |
| 7 | 3,938 |
| 8 | 26,194 |
| 9 | 186,042 |
| 10 | 1,395,008 |
| 11 | 10,948,768 |
| 12 | 89,346,128 |
| 13 | 754,062,288 |
| 14 | 6,553,942,722 |
| 15 | 58,457,558,394 |
| 16 | 533,530,004,810 |
| 17 | 4,970,471,875,914 |
| 18 | 47,169,234,466,788 |
| 19 | 455,170,730,152,340 |
| 20 | 4,459,456,443,328,824 |
| 21 | 44,300,299,824,885,392 |
| 22 | 445,703,524,836,260,400 |
| 23 | 4,536,891,586,511,660,256 |
| 24 | 46,682,404,846,719,083,048 |
| 25 | 485,158,560,873,624,409,904 |
| 26 | 5,089,092,437,784,870,584,576 |
| 27 | 53,845,049,871,942,333,501,408 |

We define $k$-clumped permutations in Section 2. For now, to place the 2-clumped permutations in context, we note that the 1-clumped permutations are the *twisted Baxter permutations*, which are in bijection with the better-known *Baxter permutations*. Baxter permutations are also relevant to the combinatorics of rectangulations. Indeed, Baxter permutations are in bijection [1,19] with the *mosaic floorplans* considered in the VLSI (Very Large Scale Integration) circuit design literature [13]. Mosaic floorplans are certain equivalence classes of generic rectangulations. (A similar result linking equivalence classes of generic rectangulations to pattern-avoiding permutations is given in [4].) In light of results of [2], the bijection from Baxter permutations to mosaic floorplans can be rephrased as a bijection to a subclass of the generic rectangulations that we call *diagonal rectangulations*, which figure prominently in this paper.

The symbol $G_n$ will denote the set of 2-clumped permutations. Let $\text{gRec}_n$ be the set of generic rectangulations with $n$ rectangles. The bijection from $G_n$ to $\text{gRec}_n$ is defined as the restriction of a map $\gamma : S_n \to \text{gRec}_n$. We show that $\gamma$ is surjective and that its fibers are the congruence classes of a lattice congruence on the weak order on $S_n$. We do not prove directly that the fibers of $\gamma$ define a congruence. Instead, we recognize the fibers as the classes of a congruence arising as one case of a construction from [17], where lattice congruences on the weak order are used to construct sub Hopf algebras of the Malvenuto–Reutenauer Hopf algebra of permutations. The results of [17] show that the 2-clumped permutations are a set of congruence class representatives. Thus the restriction of $\gamma$ is a bijection from $G_n$ to $\text{gRec}_n$.

*Note added in proof*

After this paper was accepted, the author became aware of a substantial literature studying generic rectangulations under the name *rectangular drawings*. This literature includes some results on asymptotic enumeration as well as computations of the exact cardinality of $\text{gRec}_n$ for many values of $n$. See, for example, [3,11,14]. In particular, the main result of this paper answers an open question posed in [3, Section 5].