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Classification of nonorientable regular embeddings of Hamming graphs☆

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ABSTRACT

By a regular embedding of a graph K in a surface we mean a 2-cell embedding of K in a compact connected surface such that the automorphism group acts regularly on flags. In this paper, we classify the nonorientable regular embeddings of the Hamming graph $H(d, n)$. We show that there exists such an embedding if and only if $n = 2$ and $d = 2$, or $n = 3$ or 4 and $d \geq 1$, or $n = 6$ and $d = 1$ or 2. We also give constructions and descriptions of these embeddings.

1. Introduction

A *map* $\mathcal{M}$ is a 2-cell embedding of a graph K in a compact, connected surface S. An *automorphism* of $\mathcal{M}$ is a permutation of its *flags* (mutually incident vertex–edge–face triples) which preserves their relations of having a vertex, edge or face in common; it therefore induces an automorphism of K which extends to a self-homeomorphism of S. The group $G := \text{Aut}(\mathcal{M})$ of all automorphisms of $\mathcal{M}$ acts semi-regularly on its flags, so $|G| \leq 4|E|$, where E is the set of edges. If this bound is attained then G acts regularly on the flags, and $\mathcal{M}$ is called a *regular* map. Equivalently, $\mathcal{M}$ is regular if and only if there are three involutions λ, ρ and τ in G, each fixing a distinct pair of elements v, e, f of some flag (v, e, f); in this case we have $G = \langle \lambda, \rho, \tau \rangle$. In what follows we shall assume that $\mathcal{M}$ is a regular map, with λ fixing e and f, and ρ fixing v and f, so τ fixes v and e. We call such a triple (λ, ρ, τ) an *admissible triple*.

Since $\tau\lambda = \lambda\tau$ the stabilizer $G_e = \langle \lambda, \tau \rangle$ in G of e is a dihedral group of order 4, that is, a Klein 4-group, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Similarly, the stabilizers $G_v = \langle \rho, \tau \rangle$ and $G_f = \langle \lambda, \rho \rangle$ of v and f are

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dihedral groups of orders $2m$ and $2n$, where m is the common valency of the vertices of $\mathcal{M}$, that is, the order of $\rho\tau$, and n is the covalency (the number of edges of each face), equal to the order of $\lambda\rho$.

When a regular map $\mathcal{M}$ is represented in this way by a triple of involutions λ, ρ, τ, we write $\mathcal{M} = \mathcal{M}(\lambda, \rho, \tau)$. Two regular maps $\mathcal{M}(\lambda, \rho, \tau)$ and $\mathcal{M}'(\lambda', \rho', \tau')$ with underlying graphs K and K' are *isomorphic* if there is a graph isomorphism $\psi : K \to K'$ such that $\psi^{-1}\lambda\psi = \lambda'$, $\psi^{-1}\rho\psi = \rho'$ and $\psi^{-1}\tau\psi = \tau'$. A detailed explanation of the above representation of regular maps can be found in [3, Theorem 3]. The basic theory of regular maps, as well as other relevant information, can also be found in [7,8,11–13].

If a regular map $\mathcal{M}$ is obtained from an embedding $i : K \to S$ of a graph K in a surface S, we say that i is a *regular embedding* of K. The surface S underlying $\mathcal{M}$ is nonorientable if and only if there is a cycle C in K with a neighbourhood in S homeomorphic to a Möbius band. Such a cycle will be called *orientation-reversing*. A regular map $\mathcal{M}$ is nonorientable if and only if its automorphism group G is generated by $R := \rho\tau$ and the involution $L := \lambda\tau = \tau\lambda$. In particular, if there is an orientation-reversing cycle C of length l in $\mathcal{M}$ then there is an associated relation of the form $LR^{m_1}LR^{m_2}\cdots LR^{m_l} = \tau$ in G. Conversely, the existence of such a relation in G implies that $\mathcal{M}$ is nonorientable. We call such a triple (λ, ρ, τ) an *nonorientable admissible triple*.

There are only a few families of graphs for which a complete classification of their nonorientable regular embeddings is known. Such embeddings of complete graphs K_n have been classified by James [4] and Wilson [15]: these exist if and only if n is 3, 4 or 6. Nedela and the second author [10] have shown the nonexistence of a nonorientable regular embedding of the n-dimensional cube graph Q_n for all n except $n = 2$. In contrast with all other known cases, the complete bipartite graph $K_{n,n}$ has a nonorientable regular embedding for infinitely many values of n, as shown by Kwak and the second author [9]: in fact, such an embedding exists if and only if $n \equiv 2 \mod (4)$ and all odd prime divisors of n are congruent to $\pm 1 \mod (8)$.

A map $\mathcal{M}$ is *orientably regular* if the underlying surface is orientable and the orientation-preserving subgroup $\mathrm{Aut}^+(\mathcal{M})$ of $\mathrm{Aut}(\mathcal{M})$ acts regularly on the arcs (directed edges) of $\mathcal{M}$. The first author [6] has recently obtained a classification of such embeddings of Hamming graphs $H(d, n)$, and this includes a classification of their orientable regular embeddings (called *reflexible* embeddings there). In this paper, we aim to classify the nonorientable regular embeddings of Hamming graphs. Our main result is the following theorem:

Theorem 1.1. *There exists a nonorientable regular embedding of the Hamming graph $H(d, n)$ if and only if either*

(1) $n = 2$ *and* $d = 2$, *or*
(2) $n = 3$ *or* 4 *and* $d \geq 1$, *or*
(3) $n = 6$ *and* $d = 1$ *or* 2.

In cases (1) *and* (2), *the embedding of $H(d, n)$ is unique up to isomorphism, whereas in case* (3), *there are two such embeddings for the graph $H(d, n)$.*

Further information about each of these maps, namely its type, genus and automorphism group, is given in Section 2.

This paper is organized as follows. In Section 2 we construct and describe some examples of nonorientable regular embeddings of $H(d, n)$, and in Section 3 we classify all such embeddings by showing that each of them is isomorphic to one of these examples.

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2. Construction of nonorientable Hamming maps

The *Hamming graph $H(d, n)$* is the Cartesian product of d cliques of size n. Specifically, we define $H(d, n)$ to have vertex set $V = [n]^d$ where $[n] = \{0, 1, \ldots .n - 1\}$ for some $n \geq 2$, with two vertices $u = (u_i)$ and $v = (v_i)$ adjacent if and only if $u_i = v_i$ for all except one value of i.

The automorphism group $\mathrm{Aut}(H(d, n))$ of this graph is the wreath product $S_n \wr S_d$ of the symmetric groups S_n and S_d. This is a semidirect product of a normal subgroup $S_n \times S_n \times \cdots \times S_n$, whose ith direct