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Ramsey-type constructions for arrangements of segments

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ABSTRACT

Improving a result of Károlyi, Pach and Tóth, we construct an arrangement of n segments in the plane with at most $n^{\log 8/\log 169}$ pairwise crossing or pairwise disjoint segments. We use the recursive method based on flattenable arrangements which was established by Larman, Matoušek, Pach and Törőcsik.

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1. Introduction

An *arrangement of segments* is a finite set of compact straight-line segments in the plane in general position (i.e., no three endpoints are collinear). We study the following Ramsey-type problem [6]: what is the largest number $r(k)$ such that there exists an arrangement of $r(k)$ segments with at most k pairwise crossing and at most k pairwise disjoint segments?

Larman et al. [6] proved that $k^5 \geq r(k) \geq k^{\log 5/\log 2} > k^{2.3219}$. The upper bound has remained unchanged since then. Károlyi et al. [3] improved the lower bound to $r(k) \geq k^{\log 27/\log 4} > k^{2.3774}$.

We improve the construction for the lower bound even further and prove the following theorem.

Theorem 1. *For infinitely many positive integers k, there exists an arrangement of $k^{\log 169/\log 8} > k^{2.4669}$ segments with at most k pairwise crossing and at most k pairwise disjoint segments.*

Similar questions were studied by Fox et al. [2] for string graphs, a class of graphs generalizing intersection graphs of segments. They proved, as a consequence of a stronger result, that for each positive integer k there is a constant $c(k) > 0$ such that in any system of n curves in the plane where every two curves intersect in at most k points, there is a subset of $n^{c(k)}$ curves that are pairwise disjoint or pairwise crossing.

In an extended version of this paper [5], we also show examples of arrangements of segments that cannot be flattened.

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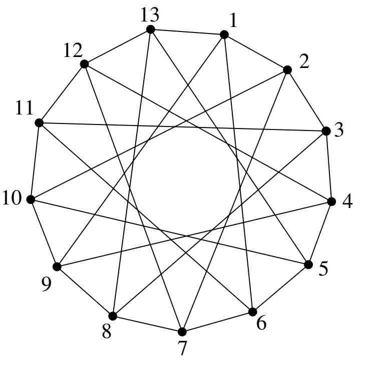

Fig. 1. A Cayley graph $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$.

2. Proof of Theorem 1

Both previous constructions for the lower bound [3,6] use the same approach. The starting configuration is an arrangement M_0 of n_0 segments with at most k_0 pairwise crossing or pairwise disjoint segments. In the i-th step, an arrangement M_i of n_0^{i+1} segments is constructed from the arrangement M_{i-1} by replacing each of its segments by a flattened copy (a precise definition will follow) of M_0, which acts as a "thick segment". Then two segments from different copies of M_0 cross if and only if the two corresponding segments in M_{i-1} cross. Our new arrangement M_i has then at most k_0^{i+1} pairwise crossing or pairwise disjoint segments. This gives a lower bound $r(k) \geq k^{\log n_0 / \log k_0}$ for infinitely many values of k.

We improve the construction by making a better starting arrangement. Unlike the previous constructions, our basic pieces will be arrangements with different maximal numbers of pairwise crossing and pairwise disjoint segments. By putting them together, we obtain our starting arrangement M_0.

Let $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ denote the Cayley graph of the cyclic group $\mathbb{Z}_{13}$ corresponding to the generators 1 and 5. That is, $V(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{1, 2, \ldots, 13\}$ and $E(\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)) = \{\{i, j\}; 1 \leq i < j \leq 13, (j - i) \in \{1, 5, 8, 12\}\}$. See Fig. 1.

Lemma 2. *The graph* $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ *contains no clique of size 3 and no independent set of size 5.*

Proof. Suppose that $a < b < c$ are three vertices of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$ inducing a clique. Then the numbers $k = c - a$, $l = c - b$ and $m = b - a$ belong to the set $\{1, 5, 8, 12\}$, but this set contains no triple k, l, m satisfying the equation $k = l + m$; a contradiction.

Now suppose that $A = \{a < b < c < d < e\}$ is an independent set of $\mathrm{Cay}(\mathbb{Z}_{13}; 1, 5)$. By the pigeon-hole principle, A contains two vertices with difference 2 (modulo 13). Thus, we can, without loss of generality, assume that $a = 1$ and $b = 3$. It follows that $\{c, d, e\} \subseteq \{5, 7, 10, 12\}$. But A cannot contain both 5 and 10, neither both 7 and 12. Hence $|A \cap \{5, 7, 10, 12\}| \leq 2$; a contradiction. □

A (k, l)-*arrangement* is an arrangement of segments with at most k pairwise crossing and at most l pairwise disjoint segments.

An *intersection graph* $G(M)$ of an arrangement M is a graph whose vertices are the segments of M and two vertices are joined by an edge if and only if the corresponding segments intersect.

An arrangement M of segments is *flattenable* if for every $\varepsilon > 0$ there is an arrangement M_ε with $G(M_\varepsilon) = G(M)$ and two discs D_1, D_2 of radius ε whose centers are at unit distance, such that each segment from M_ε has one endpoint in D_1 and the second endpoint in D_2. A *flattened copy* of M is the arrangement M_ε with sufficiently small ε.

The key result is the following lemma.

Lemma 3. (1) *There exists a flattenable* $(2, 4)$*-arrangement of 13 segments.*
(2) *There exists a flattenable* $(4, 2)$*-arrangement of 13 segments.*

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